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DISQUISITIONES QUATUOR

AD

THEORIAM FUNCTIONUM ANALYTICARUM

PERTINENTES

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6.

DISQUISITIONES QUAE JON

THEORIAM FUNCTIONUM ANALYTICARUM

AD THEOREMA CAUCHYI DE INTEGRATIONE
FUNCTIO-
NUM COMPLEXARUM



IN PARTI

EX OFFICIO ACADEMIAE SCIENTIARUM

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Disquisitio prima

de

functionibus $\text{Sin.}x$, $\text{Cos.}x$ $\frac{e^x + e^{-x}}{2}$ in factores resolvendis et
de summis serierum reciprocarum inveniendis.

§. 1.

Functiones $\text{Sin.}x$ et $\text{Cos.}x$ in factores resolvi posse Viri Illustres Joannes Bernoulli et Eulerus primi docuerunt. (J. Bernoulli opp. tom. IV. n. 152, Comment. Ac. Petr. tom. VIII. aa. 1734, 35) Methodus, qua usi sunt, hujus propositionis deducendae ob eximiam simplicitatem et brevitatem fere in omnibus libris de calculo sublimiori tractantibus recepta est. Principia tamen, quibus haec deductio superstructa est, non satis certa et dilucida sunt. Quam ob rem verisimiliter Eulerus istud theorema, quod, ut omnia a Viro Egregio tractata, e manibus ejus plurimarum novarum veritatum fertilissimum evasit, de integro aggressus (Intr. in anal. inf. L. I. §. 158) alio modo notione infiniti quantitatunque imaginariarum superstructo demonstravit. Celeb. l'Huilier, vir de principiis calculi sublimioris rite stabiliendis optime meritus, vestigiis Euleri ingressus novam hujus theorematis demonstrationem ab obscura notione infiniti liberatam methodoque exhaustionum conditam conatus est.

(Princ. calc. diff. p. 120) Quamvis ejus demonstratio duabus prioribus praeferri mereatur, nihilominus eum, qui methodo veterum demonstrandi imbutus est, si rem penitus examinaverit, non effugiet quod illius methodi rigorem et diluciditatem, cujus haec propositio capax est, non attigerit. Tentamen ergo nostrum ea, quae huic demonstrationi adhuc deesse videntur, explendi res non prorsus utilitatis expers et nullius pretii censum iri speramus.

§. 2.

Antequam nostrum propositum adgrediamur, propositiones quasdam ex theoria exhaustionum sumtas et alias, quibus demonstratio theorematis nostri superstruenda est, praemitemur; demonstrandis iis autem, cum res nulli difficultati obnoxia sit, non immorabimur.

I. Def. Si quantitas mutabilis semper crescens aut decrescens quantitati constanti ita appropinquat, ut differentia inter hanc et illam quacunque quantitate data minor fieri possit, haec quantitas constans illius quantitatis mutabilis limes nominatur.

II. Theor. Quantitas mutabilis, quae semper crescit aut decrescit quacunque autem quantitate data respective major seu minor fieri nequit, limitem habet.

III. Theor. Duae quantitates mutabiles, quarum altera crescit, dum altera decrescit, et quarum differentia quacunque quantitate minor fieri potest, limitem communem habent; et quantitas constans semper alterâ major et alterâ minor hic limes erit.

IV. Theor. Si duarum quantitatum simul crescentium aut decrescentium, quarum differentia quacunque quantitate data minor fieri potest una limite gaudet, altera etiam eundem limitem habebit.

V. Theor. Summae, differentiae, producti aut quotientis duarum quantitatum mutabilium limitibusque gaudentium, semper crescentis aut decrescentis limes est aequalis summae, differentiae, producti aut quotienti limitum harum duarum quantitatum.

VI. Theor. Denotando brevitatis gratia $(1 - \frac{1}{m})$ $(1 - \frac{2}{m})$ $(1 - \frac{3}{m})$ - - - $(1 - \frac{r}{m})$ per (m, r) et $\frac{x}{1.2.3\dots r}$ per x_c^r facile ex theoremate binomiali, pro quocunque numero integro positivo m sequitur, fore

$$(1 + \frac{x}{m})^m = 1 + x + (m, 1) x_c^2 + (m, 2) x_c^3 + (m, 3) x_c^4 \dots + (m, m-1) x_c^m$$

$$(1 - \frac{x}{m})^m = 1 - x + (m, 1) x_c^2 - (m, 2) x_c^3 + (m, 3) x_c^4 \dots \pm (m, m-1) x_c^m$$

adeoque

$$\frac{(1 + \frac{x}{m})^m - (1 - \frac{x}{m})^m}{2} = x + (m, 2) x_c^3 + (m, 4) x_c^5 \dots + (m, \mu) x_c^{\mu+1}$$

$$\frac{(1 + \frac{x}{m})^m + (1 - \frac{x}{m})^m}{2} = 1 + (m, 1) x_c^2 + (m, 3) x_c^4 \dots + (m, \nu) x_c^{\nu+1}$$

ubi $\mu = m-2$, aut $m-1$, et $\nu = m-1$ aut $m-2$, prout m sit numerus par vel impar.

VII. Simili modo e notis expressionibus pro sinu et cosinu cujusdam multipli arcus deducitur

$$\frac{\text{Sin. } m \text{ Arc. tan. } \frac{x}{m}}{(\text{Cos. Arc. tan. } \frac{x}{m})^m} = x - (m, 2) x_c^3 + (m, 4) x_c^5 - (m, 6) x_c^7 \dots \pm (m, \mu) x_c^{\mu}$$

$$\frac{\text{Cos. } m \text{ Arc. tan. } \frac{x}{m}}{(\text{Cos. } m \text{ Arc. tan. } \frac{x}{m})^m} = 1 - (m, 1) x_c^2 + (m, 3) x_c^4 - (m, 5) x_c^6 \dots \mp (m, \nu) x_c^{\nu}$$

VIII. Ex his aequationibus omni cum rigore theoriae aequationum ope sequitur, π circumferentiam circuli, cujus radius $= 1$, denotante,

$$(A) \quad \frac{\text{Sin. } m \text{ Arc. tan. } \frac{x}{m}}{(\text{Cos. Arc. tan. } \frac{x}{m})^m} = x \left(1 - \frac{x^2}{m^2 \tan^2 \frac{\pi}{2m}} \right) \left(1 - \frac{x^2}{m^2 \tan^2 \frac{3\pi}{2m}} \right) \dots \left(1 - \frac{x^2}{m^2 \tan^2 \frac{\mu\pi}{2m}} \right)$$

$$(B) \quad \frac{\text{Cos. } m \text{ Arc. tan. } \frac{x}{m}}{(\text{Cos. Arc. tan. } \frac{x}{m})^m} = \left(1 - \frac{x^2}{m^2 \tan^2 \frac{\pi}{2m}} \right) \left(1 - \frac{x^2}{m^2 \tan^2 \frac{3\pi}{2m}} \right) \dots \left(1 - \frac{x^2}{m^2 \tan^2 \frac{\nu\pi}{2m}} \right)$$

$\frac{1}{2}m-1$ aut $\frac{m-1}{2}$ per μ , et $m-1$ aut m per ν denotando, prout sit m numerus par vel impar.

VIII. Ex propositionibus VI, VII, VIII. nullo negotio sequuntur aequationes

$$(C) \quad \frac{(1 + \frac{x}{m})^m - (1 - \frac{x}{m})^m}{2} = x \left(1 + \frac{x^2}{m^2 \tan^2 \frac{\pi}{2m}} \right) \left(1 + \frac{x^2}{m^2 \tan^2 \frac{3\pi}{2m}} \right) \dots \left(1 + \frac{x^2}{m^2 \tan^2 \frac{\mu\pi}{2m}} \right)$$

$$(D) \quad \frac{(1 + \frac{x}{m})^m + (1 - \frac{x}{m})^m}{2} = \left(1 + \frac{x^2}{m^2 \tan^2 \frac{\pi}{2m}} \right) \left(1 + \frac{x^2}{m^2 \tan^2 \frac{3\pi}{2m}} \right) \dots \left(1 + \frac{x^2}{m^2 \tan^2 \frac{\nu\pi}{2m}} \right)$$

Quas duas aequationes Celeb. l'Huilier ex theoremate Cotesiano deduxit, cujus ipsius tamen demonstratio, si quantitates imaginarias adhibere nolueris, satis prolixa est (Joann. Bernoulli. opp. t. 4. n. 160) et facilius ex aequationibus (C), (D), substituendo x loco $\frac{x}{m}$, mutatis derivari potest.

X. Aequatio (A), $\sqrt{\frac{x}{1 - \frac{x^2}{m^2}}}$ loco ipsius x substituto, transit m numerum imparem denotante, in

$$(E) \quad \text{Sin. } m \text{ Arc. Sin. } \frac{x}{m} = x \left(1 - \frac{x^2}{m^2 \text{ Sin. }^2 \frac{\pi}{m}}\right) \left(1 - \frac{x^2}{m^2 \text{ Sin. }^2 \frac{2\pi}{m}}\right) \left(1 - \frac{x^2}{m^2 \text{ Sin. }^2 \frac{3\pi}{m}}\right) \dots \left(1 - \frac{x^2}{m^2 \text{ Sin. }^2 \frac{\frac{1}{2}(m-1)\pi}{m}}\right)$$

et aequatio (B), eadem substitutione facta, et m numerum parem denotante fit:

$$(F) \quad \text{Cos. } m \text{ Arc. Sin. } \frac{x}{m} = \left(1 - \frac{x^2}{m^2 \text{ Sin. }^2 \frac{\pi}{2m}}\right) \left(1 - \frac{x^2}{m^2 \text{ Sin. }^2 \frac{3\pi}{2m}}\right) \left(1 - \frac{x^2}{m^2 \text{ Sin. }^2 \frac{5\pi}{2m}}\right) \dots \left(1 - \frac{x^2}{m^2 \text{ Sin. }^2 \frac{(m-1)\pi}{2m}}\right)$$

XI. Simili modo, substituendo $\sqrt{\frac{x}{1 + \frac{x^2}{m^2}}}$ loco ipsius x , aequatio (C) pro m impari fit:

$$(G) \quad \frac{\left(\sqrt{1 + \frac{x^2}{m^2}} + \frac{x}{m}\right)^m - \left(\sqrt{1 + \frac{x^2}{m^2}} - \frac{x}{m}\right)^m}{2} = x \left(1 + \frac{x^2}{m^2 \text{ Sin. }^2 \frac{\pi}{m}}\right) \left(1 + \frac{x^2}{m^2 \text{ Sin. }^2 \frac{2\pi}{m}}\right) \left(1 + \frac{x^2}{m^2 \text{ Sin. }^2 \frac{3\pi}{m}}\right) \dots \left(1 + \frac{x^2}{m^2 \text{ Sin. }^2 \frac{\frac{1}{2}(m-1)\pi}{m}}\right)$$

et pro m pari transmutatur (D) in

$$(H) \quad \frac{\left(\sqrt{1 + \frac{x^2}{m^2}} + \frac{x}{m}\right)^m + \left(\sqrt{1 + \frac{x^2}{m^2}} - \frac{x}{m}\right)^m}{2} = \left(1 + \frac{x^2}{m^2 \sin^2 \frac{\pi}{2m}}\right) \\ \left(1 + \frac{x^2}{m^2 \sin^2 \frac{3\pi}{2m}}\right) \dots \left(1 + \frac{x^2}{m^2 \sin^2 \frac{(m-1)\pi}{2m}}\right)$$

§. 3.

Lemma. Signo ϕx talem functionem ipsius x denotante, quae, existente $x = 0$, fit $= 0$ et x aucto, crescit, erit, si $y > x$, $\frac{\phi y}{y} >$ aut $< \frac{\phi x}{x}$, prout ϕx respectu ipsius x vel accelerando vel retardando crescit.

Demonst. I. Sit y respectu x quantitas rationalis, erit $y = \frac{m}{n}x$, m, n numeros positivos integros denotantibus. Statuendo $\frac{x}{n} = \xi$, $\phi \xi = \eta$, $\phi^2 \xi - \phi \xi = \eta'$, $\phi^3 \xi - \phi^2 \xi = \eta''$, etc. habebitur $\phi n \xi$ aut $\phi x = \eta + \eta' + \eta'' + \eta''' \dots \eta^{(n)}$ et $\phi y = \eta + \eta' + \eta'' + \eta''' \dots \eta^{(m)}$; erit ergo, ϕx respectu ipsius x accelerando crescente, $\eta < \eta' < \eta'' < \eta''' < \dots$ et $\frac{\eta}{\xi} < \frac{\eta + \eta'}{2\xi} < \frac{\eta + \eta' + \eta''}{3\xi} < \dots$ etc. adeoque etiam $\frac{\phi x}{x} < \frac{\phi y}{y}$. Contra ea, ϕx retardando aucta, erit $\eta > \eta' > \eta'' > \eta''' < \dots$ et $\frac{\eta}{\xi} > \frac{\eta + \eta'}{2\xi} > \frac{\eta + \eta' + \eta''}{3\xi} > \dots$ etc. et $\frac{\phi x}{x} > \frac{\phi y}{y}$.

II. Si y respectu x est quantitas irrationalis, et, ϕx accelerando crescente, pro $y > x$, non esset $\frac{\phi y}{y} > \frac{\phi x}{x}$, foret $\frac{\phi y}{y} =$ aut $< \frac{\phi x}{x}$ et quia $y = \frac{m}{n}x + \omega$, ubi n numerum quemcunque positivum integrum et ω quantitatem minorem quam $\frac{x}{n}$ aut ξ denotant, deberet esse $\frac{\phi(m\xi + \omega)}{m\xi + \omega} =$ aut $< \frac{\phi n\xi}{n\xi}$, ad eoque etiam $\frac{\phi(m\xi)}{m\xi} - \frac{\phi(m\xi + \omega)}{m\xi + \omega} =$ aut $> \frac{\phi \cdot m\xi}{m\xi} - \frac{\phi \cdot n\xi}{n\xi}$. Cum autem $\frac{\phi \cdot m\xi}{m\xi} - \frac{\phi \cdot n\xi}{n\xi}$, si $m > n$, sit quantitas data positiva, etiam $\frac{\phi m\xi}{m\xi} - \frac{\phi(m\xi + \omega)}{m\xi + \omega}$ quantitas data positiva sit necesse est. Statuendo $\phi(m\xi + \omega) = \phi m\xi + \Omega$, erit Ω quantitas, quae ω decrescente, quantitate quacunque data minor fieri potest, et quia $\frac{\phi(m\xi)}{m\xi} - \frac{\phi(m\xi + \omega)}{m\xi + \omega} = \frac{\omega \phi m\xi - m\xi \cdot \Omega}{m\xi \cdot (m\xi + \omega)}$, manifestum est, hanc differentiam, quae quantitas debet esse data, quacunque quantitate minorem fieri posse, q. e. a.

Si, ϕx retardando crescente, et existente $y > x$, $\frac{\phi y}{y}$ non esset $< \frac{\phi x}{x}$, foret $\frac{\phi(m\xi + \omega)}{m\xi + \omega} =$ aut $> \frac{\phi n\xi}{n\xi}$ et $\frac{\phi(m\xi + \omega)}{m\xi + \omega} - \frac{\phi m\xi}{m\xi} =$ aut $> \frac{\phi n\xi}{n\xi} - \frac{\phi m\xi}{m\xi}$, id quod fieri nequit, quia $\frac{\phi(m\xi + \omega)}{m\xi + \omega} - \frac{\phi m\xi}{m\xi}$ quacunque quantitate fieri posse simulque quantitati datae aequalis aut major esse oportet.

§. 4.

Coroll. Ex hoc theoremate generali sequitur, pro quibuscunque valoribus positivis x , $y < \frac{1}{2}\pi$, fore, existente $y > x$, $\frac{\sin x}{x} >$

$\frac{\sin. y}{y}$ et $\frac{\tan. x}{x} < \frac{\tan. y}{y}$. Etenim, si $\sin \zeta = \eta'$, $\sin. 2 \zeta = \sin. \zeta = \eta''$,
 $\sin. 3 \zeta = \sin. 2 \zeta = \eta'''$, etc. erit $\eta' = 2 \cos. \frac{\zeta}{2} \sin. \frac{\zeta}{2}$, $\eta'' = 2 \cos. \frac{3\zeta}{2} \sin. \frac{\zeta}{2}$,
 $\eta''' = 2 \cos. \frac{5\zeta}{2} \sin. \frac{\zeta}{2}$, etc. ergo $\eta' > \eta'' > \eta''' > \text{etc.}$ Retardando
 igitur $\sin. x$, x aucto, crescit; itaque $\frac{\sin. x}{x} > \frac{\sin. y}{y}$. Sin autem
 $\eta' = \tan. \zeta$, $\eta'' = \tan. 2 \zeta = \tan. \zeta$, $\eta''' = \tan. 3 \zeta = \tan. 2 \zeta$, etc.
 erit $\eta' = \frac{\sin. \zeta}{\cos. \zeta}$, $\eta'' = \frac{\sin. \zeta}{\cos. \zeta \cos. 2\zeta}$, $\eta''' = \frac{\sin. \zeta}{\cos. 2\zeta \cos. 3\zeta}$, etc. adeo-
 que $\eta' < \eta'' < \eta''' < \text{etc.}$ $\tan. x$ igitur accelerando crescit et $\frac{\tan. x}{x}$
 $< \frac{\tan. y}{y}$ esse oportet.

§. 5.

Coroll. Sicut est, existente $y > x$, $\frac{\sin. x}{x} > \frac{\sin. y}{y}$ et $\frac{\tan. x}{x}$
 $< \frac{\tan. y}{y}$, ita habebitur etiam, si $y > x$, $\frac{\text{Arc. Sin. } x}{x} < \frac{\text{Arc. Sin. } y}{y}$
 et $\frac{\text{Arc. tan. } x}{x} > \frac{\text{Arc. tan. } y}{y}$; nec non oportet esse, pro $m > n$,
 $n \text{ Arc. Sin. } \frac{x}{n} > m \text{ Arc. Sin. } \frac{x}{m}$ et $n \text{ Arc. tan. } \frac{x}{n} < m \text{ Arc. tan. } \frac{x}{m}$.

§. 6.

Coroll. Cum sit, existente $y > x$, $\frac{\sin. (y-x)}{y-x} > \frac{\sin. y}{y}$ erit
 etiam $\frac{\sin. (y-x)}{\sin. y} > \frac{y-x}{y}$ eoque magis $\frac{\sin. (y-x)}{\sin. x \sin. y} > \frac{y-x}{yx}$ aut
 $\frac{1}{\tan. x} - \frac{1}{\tan. y} > \frac{1}{x} - \frac{1}{y}$ et $\frac{1}{y} - \frac{1}{\tan. y} > \frac{1}{x} - \frac{1}{\tan. x}$.

§. 7.

Praeter propositiones modo exhibitas utemur adhuc sequentibus: existente $x < y < \frac{1}{2} \pi$, erit $\text{Cos. } x - \text{Cos. } y$ et $\text{Sin. } y - \text{Sin. } x < y - x$; $\text{Cot. } x - \text{Cot. } y$ et $\text{tang. } y - \text{tang. } x > y - x$; adeoque etiam $\text{Arc. Cos. } x - \text{Arc. Cos. } y$ et $\text{Arc. Sin. } y - \text{Arc. Sin. } x > y - x$; $\text{Arc. Cot. } x - \text{Arc. Cot. } y$ et $\text{Arc. tang. } y - \text{Arc. tang. } x < y - x$.

§. 8.

Lemma. Quantitatibus duabus $\left(1 + \frac{x}{2^n}\right)^{2^n}$, et $\left(1 - \frac{x}{2^n}\right)^{-2^n}$, numero integro positivo n aucto et x quemcunque numerum positivum denotante, est limes communis $= \left[\lim. \left(1 + \frac{x}{2^n}\right)^{2^n} \right]^x$

Demonstr. 1) Est $\left(1 + \frac{x}{2^{n+1}}\right)^{2^{n+1}} = \left(1 + \frac{x}{2^n} + \frac{x^2}{2^{2n+2}}\right)^{2^n} > \left(1 + \frac{x}{2^n}\right)^{2^n}$ et $\left(1 - \frac{x}{2^{n+1}}\right)^{-2^{n+1}} = \left(1 - \frac{x}{2^n} + \frac{x^2}{2^{2n+2}}\right)^{-2^n} < \left(1 - \frac{x}{2^n}\right)^{-2^n}$; quantitas igitur $\left(1 + \frac{x}{2^n}\right)^{2^n}$, n aucto, crescit dum altera $\left(1 - \frac{x}{2^n}\right)^{-2^n}$ decrescit. Differentia autem $\left(1 - \frac{x}{2^n}\right)^{-2^n} - \left(1 + \frac{x}{2^n}\right)^{2^n}$, statuendo $2^n = m$, est $= \left(1 - \frac{x}{m}\right)^{-m} - \left(1 + \frac{x}{m}\right)^m = (\S. 2, \text{VI}) \left(1 - \frac{x}{m}\right)^{-m} \left(\frac{x^2}{m} - (m, 1) \left(\frac{x^2}{m}\right)_c + (m, 2) \left(\frac{x^2}{m}\right)_c^2 - \dots - (m, m-1) \left(\frac{x^2}{m}\right)_c^{m-1}\right)$ adeoque, existente $m > x$, $< \left(1 - \frac{x}{m}\right)^{-m} \frac{x^2}{m}$. Patet ergo, cum, $m = 2^n$ crescente, $\left(1 - \frac{x}{m}\right)^{-m}$

decreseat et $\frac{x^2}{m}$ quacunque quantitate data minor fieri possit, eo magis differentiam $\left(1 - \frac{x}{2^2}\right)^{-2n} - \left(1 + \frac{x}{2^2}\right)^{2n}$ quacunque quantitate minorem fieri posse. Unde (§. 2, III.) apparet, quantitates $\left(1 + \frac{x}{2^n}\right)^{2n}$, $\left(1 - \frac{x}{2^n}\right)^{-2n}$ limite communi gavisuras esse.

2. Quia $\left(1 + \frac{x+y}{m}\right)^m < \left(1 + \frac{x}{m}\right)^m \left(1 + \frac{y}{m}\right)^m < \left(1 - \frac{x}{m}\right)^{-m}$
 $\left(1 - \frac{y}{m}\right)^{-m} < \left(1 - \frac{x+y}{m}\right)^{-m}$ et $\left(1 - \frac{x+y}{m}\right)^{-m} - \left(1 + \frac{x+y}{m}\right)^m$
 quacunque quantitate minor fieri potest, idem eo magis pro
 $\left(1 + \frac{x}{m}\right)^m \left(1 + \frac{y}{m}\right)^m - \left(1 + \frac{x+y}{m}\right)^m$ locum habebit, quamobrem
 erit (§. 2, V.) $\lim. \left(1 + \frac{x}{m}\right)^m \lim. \left(1 + \frac{y}{m}\right)^m = \lim. \left(1 + \frac{x+y}{m}\right)^m$.
 Unde facile deducitur, r et n numeros quoscunque positivos integros denotantibus, fore $\left[\lim. \left(1 + \frac{x}{m}\right)^m\right]^r = \lim. \left(1 + \frac{rx}{m}\right)^m$ et
 $\left[\lim. \left(1 + \frac{y}{m}\right)^m\right]^n = \lim. \left(1 + \frac{ny}{m}\right)^m$ adeoque etiam $r \log. \lim. \left(1 + \frac{x}{m}\right)^m = \log. \lim. \left(1 + \frac{rx}{m}\right)^m$ et $n \log. \lim. \left(1 + \frac{y}{m}\right)^m = \log. \lim. \left(1 + \frac{ny}{m}\right)^m$. Facile autem apparet, statuendo $rx >$, aut $=$, aut $< ny$, fore etiam $\log. \lim. \left(1 + \frac{rx}{m}\right)^m$ respective $>$, $=$ aut $< \log. \lim. \left(1 + \frac{ny}{m}\right)^m$ et $r \log. \lim. \left(1 + \frac{x}{m}\right)^m >$, aut $=$, aut $< n \log. \lim. \left(1 + \frac{y}{m}\right)^m$. Hinc, ex definitione proportionis ab Eu-

clide data, $x : y = \log. \lim. \left(1 + \frac{x}{m}\right)^m : \log. \lim. \left(1 + \frac{y}{m}\right)^m$
adeoque etiam $x : 1 = \log. \lim. \left(1 + \frac{x}{m}\right)^m : \log. \lim. \left(1 + \frac{1}{m}\right)^m$
et $\log. \lim. \left(1 + \frac{x}{m}\right)^m = x \log. \lim. \left(1 + \frac{1}{m}\right)^m$ unde sequitur $\lim.$
 $\left(1 + \frac{x}{m}\right)^m = \left[\lim. \left(1 + \frac{1}{m}\right)^m\right]^x$

§. 9.

Coroll. Methodo inductionis facile demonstratur, existente w
 < 1 , esse $(1 - w)^{-m} > 1 + m w + \frac{m(m+1)}{1 \cdot 2} w^2 \dots +$
 $\frac{m(m+1)(m+2) \dots (2m-1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot m} w^m$ adeoque, denotando $\left(1 + \frac{1}{m}\right)$
 $\left(1 + \frac{2}{m}\right) \dots \left(1 + \frac{r}{m}\right)$ per $[m, r]$, erit $\left(1 - \frac{x}{m}\right)^{-m} > 1 + x + [m, 1] x_c^2$
 $+ [m, 2] x_c^3 \dots + [m, m-1] x_c^m$ et quia $\left(1 + \frac{x}{m}\right)^m = 1 + x + (m, 1) x_c^2$
 $+ (m, 2) x_c^3 \dots + (m, m-1) x_c^m$ (§. 2, VI) habebitur $\left(1 - \frac{x}{m}\right)^{-m} > 1 + x$
 $+ x_c^2 + x_c^3 \dots + x_c^m > \left(1 + \frac{x}{m}\right)^m$; itaque $\lim. \left(1 + \frac{x}{m}\right)^m = \lim.$
 $\left(1 + x + x_c^2 + \dots + x_c^m\right)$ et $\lim. \left(1 + \frac{1}{m}\right)^m = \lim. \left(1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} \dots + \frac{1}{1 \cdot 2 \cdot 3 \dots m}\right) = 2, 718281828459045 \dots$ Qui
numerus basis logarithmorum naturalium dicitur et plerumque per lit-
teram e designatur, ita ut sit $\lim. \left(1 + \frac{x}{m}\right)^m = e^x = \lim. \left(1 - \frac{x}{m}\right)^{-m}$.

§. 10.

Coroll. Quia $\lim. \left(1 - \frac{x}{2^n}\right)^{-2^n} = e^x$, erit etiam (§. 2. V.)

$\lim. \left(1 - \frac{x}{2^n}\right)^{2^n} = e^{-x}$, adeoque limites utriusque quantitatis

$\frac{\left(1 + \frac{x}{2^n}\right)^{2^n} + \left(1 - \frac{x}{2^n}\right)^{2^n}}{2}$, quae n aucto crescit (§. 2 VI), erunt

$\frac{e^x + e^{-x}}{2}$. Differentiae ergo $\frac{e^x + e^{-x}}{2} = \frac{\left(1 + \frac{x}{2^n}\right)^{2^n} + \left(1 - \frac{x}{2^n}\right)^{2^n}}{2}$,

igiturque etiam $\frac{e^x + e^{-x}}{2} = \frac{\left(1 + \frac{x}{m}\right)^m + \left(1 - \frac{x}{m}\right)^m}{2}$ quacun-

que quantitate data minores fieri possunt; et quia, m quemcunque numerum positivum integrum crescentem denotante, utraque quan-

titas $\frac{\left(1 + \frac{x}{m}\right)^m + \left(1 - \frac{x}{m}\right)^m}{2}$ augetur, erit limes ejus $= \frac{e^x + e^{-x}}{2}$.

§. 11.

Coroll. Cum sit $\lim. \left(1 + x + (m, 1)x_c^2 + \dots + (m, m-1)x_c^m\right) = \lim. \left(1 + x + x_c^2 + x_c^3 + \dots + x_c^m\right)$, nullo negotio intelligitur oportere esse etiam $\lim. \left(x + (m, 2)x_c^3 + (m, 4)x_c^5 + \dots + (m, \mu)x_c^{\mu \times 1}\right)$ aut $\frac{e^x - e^{-x}}{2} = \lim. \left(x + x_c^3 + x_c^5 + \dots + x_c^{\mu \times 1}\right)$ et $\lim. \left(1 + (m, 1)x_c^2 + (m, 3)x_c^4 + \dots + (m, \nu)x_c^{\nu \times 1}\right)$ aut $\frac{e^x + e^{-x}}{2} = \lim. \left(1 + x_c^2 + x_c^4 + \dots + x_c^{\nu \times 1}\right)$ adeoque $\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2}$ aut $e^{-x} = \lim. \left(1 - x + x_c^2 - x_c^3 + \dots - x_c^{\mu \times 1}\right) = \lim. \left(1 - x + x_c^2 - x_c^3 + \dots + x_c^{\nu \times 1}\right)$, ubi $\mu = m - 2$ aut $m - 1$, et $\nu = m - 1$, aut $m - 2$ prout est m numerus par vel impar.

§. 12.

Theorema. Denotando productum $x \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \left(1 - \frac{x^2}{16\pi^2}\right)$ per $P^{(n)}$ erit, n crescente, $\text{Lim. } P^{(n)} = \text{Sin. } x$.

Demonstr. Ex aequationibus (A) (E) et (§ 5) apparet, statuendo $m = 2n + 1$, fore

$$\frac{\text{Sin. } m \text{ Arc. tang. } \frac{x}{m}}{(\text{Cos. Arc. tang. } \frac{x}{m})^m} > P^{(n)} > \text{Sin. } m \text{ Arc. Sin. } \frac{x}{m}.$$

Est autem $\text{Sin. } m \text{ Arc. tang. } \frac{x}{m} < \text{Sin. } x < \text{Sin. } m \text{ Arc. Sin. } \frac{x}{m}$ adeoque

$$P^{(n)} - \text{Sin. } x < \frac{\text{Sin. } m \text{ Arc. tang. } \frac{x}{m}}{(\text{Cos. Arc. tang. } \frac{x}{m})^m} - \text{Sin. } m \text{ Arc. tang. } \frac{x}{m}$$

$$\sqrt{1 + \frac{x^2}{m^2}} < \left(\sqrt{1 + \frac{x^2}{m^2}}\right)^m - 1 \text{ aut } < \frac{x^2}{m} + (m, 1) \left(\frac{x^2}{m}\right)_c + (m, 2)$$

$$\left(\frac{x^2}{m}\right)_c^3 \dots + (m, m-1) \left(\frac{x^2}{m}\right)_c^m, \text{ ergo etiam } < \frac{x^2}{m} + \left(\frac{x^2}{m}\right)^3 + \left(\frac{x^2}{m}\right)^3 \dots +$$

$$\left(\frac{x^2}{m}\right)^m \text{ aut } < \frac{x^2}{m \left(1 - \frac{x^2}{m}\right)}. \text{ Hinc differentia } P^{(n)} - \text{Sin. } x < \frac{x^2}{m \left(1 - \frac{x^2}{m}\right)}$$

quacunque quantitate data minor fieri potest. Cum autem $P^{(n)}$, n

aucto, decreseat erit (§. 2, 1) $\text{lim. } P^{(n)} = \text{Sin. } x$, q. e. d.

§. 13.

Coroll. Quia (§. 11) $\text{lim. } (x + (m, 2) x_c^5 + (m, 4) x_c^5 \dots + (m, 2r) x_c^{2r+1}) = \text{lim. } (x + x_c^5 + x_c^5 \dots x_c^{2r+1})$ adeoque $(1 - (m, 2)) x_c^5 + (1 - (m, 4)) x_c^5 \dots (1 - (m, 2r)) x_c^{2r+1}$ qua-

cunque quantitate data minor fieri potest, idem pro $(1 - (m, 2))x_c^5 - (1 - (m, 4))x_c^5 + (1 - (m, 6))x_c^7 - \dots + (1 - (m, 2r))x_c^{2r+1}$ tanto magis locum habebit. Erit ergo $\lim. (x - x_c^3 + x_c^5 - \dots + x_c^{2r+1}) = \lim. (x - (m, 2)x_c^3 + (m, 4)x_c^5 - \dots + (m, 2r)x_c^{2r+1}) =$
 $\lim. \frac{\text{Sin. } m \text{ Arc. tang. } \frac{x}{m}}{(\text{Cor. Arc. tang. } \frac{x}{m})^m} = \text{Sin. } x.$

§. 14.

Theor. Statuendo $(1 - \frac{4x^2}{\pi^2})(1 - \frac{4x^2}{9\pi^2})(1 - \frac{4x^2}{25\pi^2}) \dots$
 $(1 - \frac{4x^2}{(2n-1)^2\pi^2}) = Q^{(n)}$, erit $\text{Cos. } x = \lim. Q^{(n)}$.

Demonstr. Ex aeqq. (B) (F) et (§ 5) patet, existente $m = 2n$,
 esse $\frac{\text{Cos. } m \text{ Arc. tang. } \frac{x}{m}}{(\text{Cos. Arc. tang. } \frac{x}{m})^m} > Q^{(n)} > \text{Cos. } m \text{ Arc. Sin. } \frac{x}{m}$. Simul ex aeq.

(B) et (§ 5) perspicitur $\frac{\text{Cos. } m \text{ Arc. tang. } \frac{x}{m}}{(\text{Cos. Arc. tang. } \frac{x}{m})^m}$, m aucto, decrescere,

dum $\text{Cos. } m \text{ Arc. Sin. } \frac{x}{m}$ crescit. Est autem $\frac{\text{Cos. } m \text{ Arc. tang. } \frac{x}{m}}{(\text{Cos. Arc. tang. } \frac{x}{m})^m}$

$-\text{Cos. } m \text{ Arc. tang. } \frac{x}{m} < \sqrt{1 + \frac{x^2}{m^2}} - 1 < \frac{x^2}{m(1 - \frac{x^2}{m^2})}$ (§ 12) et § 7)

$\text{Cos. } m \text{ Arc. tang. } \frac{x}{m} - \text{Cos. } m \text{ Arc. Sin. } \frac{x}{m} < \sqrt{\frac{x^2}{1 - \frac{x^2}{m^2}}} - x$

$$< \frac{x}{1 - \frac{x^2}{m^2}} - x \text{ adeoque } \frac{\text{Cos. } m \text{ Arc. tang. } \frac{x}{m}}{\left(\text{Cos. Arc. tang. } \frac{x}{m}\right)^m} - \text{Cos. } m \text{ Arc. Sin. } \frac{x}{m}$$

$$< \frac{x^2}{m - x^2} + \frac{x^3}{m^2 - x^2} \text{ et quacunque quantitate data minor fieri po-}$$

$$\text{test. Sequitur ergo (§. 2, III, IV) quantitates duas } \frac{\text{Cos. } m \text{ Arc. tang. } \frac{x}{m}}{\left(\text{Cos. Arc. tang. } \frac{x}{m}\right)^m},$$

Cos m Arc. Sin. $\frac{x}{m}$ limite communi gaudere et, quia quantitas semper crescens $Q^{(n)}$ et Cos x . inter has duas quantitates mutabiles positae sunt, hunc limitem esse $= \lim Q^{(n)} = \text{Cos. } x$, q. e. d.

§. 15.

Coroll. Simili modo atque (§. 13) ostendimus, esse Sin x
 $= \lim. (x - x_c^3 + x_c^5 - \dots + x_c^{2r-1})$, perspicitur etiam fore Cos. x
 $= \lim. (1 - x_c^2 + x_c^4 - x_c^6 + \dots + x_c^{2r})$.

§. 16.

Theorema. Statuendo $x \left(1 + \frac{x^2}{\pi^2}\right) \left(1 + \frac{x^2}{4\pi^2}\right) \left(1 + \frac{x^2}{9\pi^2}\right) \dots$
 $\left(1 + \frac{x^2}{n^2\pi^2}\right) = P_n$ erit, n crescente, $\lim. P_n = \frac{e^x - e^{-x}}{2}$.

Demonstr. Ex aequationibus (C) (G) (§. 2. VIII, XI) facile
 apparet, si $m = 2n + 1$, fore $\frac{\left(1 + \frac{x}{m}\right)^m - \left(1 - \frac{x}{m}\right)^m}{2} < P_n$

$$< \frac{\left(\sqrt{1 + \frac{x^2}{m^2}} + \frac{x}{m}\right)^m - \left(\sqrt{1 + \frac{x^2}{m^2}} - \frac{x}{m}\right)^m}{2} \text{ Est autem } \left(\sqrt{1 + \frac{x^2}{m^2}} + \frac{x}{m}\right)^m$$

$$\begin{aligned}
&= \left(\sqrt{1 + \frac{x^2}{m^2}} - \frac{x}{m} \right)^{-m} < \left(1 - \frac{x}{m} \right)^{-m} \text{ et } \left(\sqrt{1 + \frac{x^2}{m^2}} - \frac{x}{m} \right)^m \\
&> \left(1 - \frac{x}{m} \right)^m \text{ adeoque etiam } \frac{\left(1 + \frac{x}{m} \right)^m - \left(1 - \frac{x}{m} \right)^m}{2} < P_n \\
&< \frac{\left(1 - \frac{x}{m} \right)^{-m} - \left(1 - \frac{x}{m} \right)^m}{2} \text{ et } P_n - \frac{\left(1 + \frac{x}{m} \right)^m - \left(1 - \frac{x}{m} \right)^m}{2} \\
&< \frac{\left(1 - \frac{x}{m} \right)^{-m} - \left(1 + \frac{x}{m} \right)^m}{2}, \text{ quam quantitatem quacunque data}
\end{aligned}$$

minorem fieri posse (§. 9) demonstravimus. Erit ergo, quia P_n , n aucto, semper crescit (§. 2, IV)

$$\lim. P_n = \lim. \frac{\left(1 + \frac{x}{m} \right)^m - \left(1 - \frac{x}{m} \right)^m}{2} = \frac{e^x - e^{-x}}{2} \text{ q. e. d.}$$

§. 17.

Theorema. Denotando $\left(1 + \frac{4x^2}{\pi^2} \right) \left(1 + \frac{4x^2}{3\pi^2} \right) \left(1 + \frac{4x^2}{5\pi^2} \right) \dots$
 $\left(1 + \frac{4x^2}{(2n-1)\pi^2} \right)$ per Q_n erit $\lim. Q_n = \frac{e^x + e^{-x}}{2}$.

Demonstr. Ex aeqq. (D) (H) perspicitur esse, si $m = 2n$

$$\frac{\left(1 + \frac{x}{m} \right)^m + \left(1 - \frac{x}{m} \right)^m}{2} < Q_n < \frac{\left(\sqrt{1 + \frac{x^2}{m^2}} + \frac{x}{m} \right)^m + \left(\sqrt{1 + \frac{x^2}{m^2}} - \frac{x}{m} \right)^m}{2}$$

ergo etiam, cum sit $\left(\sqrt{1 + \frac{x^2}{m^2}} + \frac{x}{m} \right)^m < \left(1 - \frac{x}{m} \right)^{-m}$ et

$$\left(\sqrt{1 + \frac{x^2}{m^2}} - \frac{x}{m} \right)^m = \left(\sqrt{1 + \frac{x^2}{m^2}} + \frac{x}{m} \right)^{-m} < \left(1 + \frac{x}{m} \right)^{-m},$$

$$\frac{\left(1 + \frac{x}{m} \right)^m + \left(1 - \frac{x}{m} \right)^m}{2} < Q_n < \frac{\left(1 - \frac{x}{m} \right)^{-m} + \left(1 + \frac{x}{m} \right)^{-m}}{2}$$

Itaque erit $Q_n = \frac{\left(1 + \frac{x}{m}\right)^m + \left(1 - \frac{x}{m}\right)^m}{2} > \frac{\left(1 - \frac{x}{m}\right)^{-m} + \left(1 + \frac{x}{m}\right)^{-m}}{2}$
 $= \frac{\left(1 + \frac{x}{m}\right)^m + \left(1 - \frac{x}{m}\right)^m}{2}$ aut $< \frac{1}{2} \left[\left(1 - \frac{x}{m}\right)^{-m} - \left(1 + \frac{x}{m}\right)^m \right]$
 $\times \left(1 + \frac{\left(1 - \frac{x}{m}\right)^m}{1 + \frac{x}{m}}\right) < \left[\left(1 - \frac{x}{m}\right)^{-m} - \left(1 + \frac{x}{m}\right)^m \right]$ et quantitate
 quacunque data minor fieri potest. Quantitas ergo mutabilis Q_n
 n aucto, semper crescens cum quantitate $\frac{\left(1 + \frac{x}{m}\right)^m + \left(1 - \frac{x}{m}\right)^m}{2}$, limite
 communi gaudet et erit $\lim. Q_n = \frac{e^x + e^{-x}}{2}$. q. e. d.

§. 18.

Theorema. Si ponatur $\frac{1}{x} = \frac{1}{\pi - x} + \frac{1}{\pi + x} = \frac{1}{2\pi - x} +$
 $\frac{1}{2\pi + x} \dots = \frac{1}{n\pi - x} = \varphi(x, n)$ et $\varphi(x, n) + \frac{1}{n\pi + x} = \psi(x, n)$
 erit, n aucto, $\lim. \varphi(x, n) = \lim. \psi(x, n) = \text{Cot. } x$.

Demonstr. Functiones $\varphi(x, n)$, $\psi(x, n)$ limitem communem
 habere, sponte apparet, quia, n aucto, $\varphi(x, n)$ crescit, $\psi(x, n)$ de-
 crescit et differentia earum $\frac{1}{n\pi + x}$ quac. quant. data minor fieri
 potest. Omni autem cum rigore demonstratur (Euleri Introd.

§. 251) esse $2n \text{ Cot. } x = \text{Cot. } \frac{x}{2n} - \text{Cot. } \frac{\pi - x}{2n} + \text{Cot. } \frac{\pi + x}{2n} -$
 $\text{Cot. } \frac{2\pi - x}{2n} \dots - \text{Cot. } \frac{n\pi - x}{2n}$; $(2n + 1) \text{ Cot. } x = \text{Cot. } \frac{x}{2n + 1}$
 $- \text{Cot. } \frac{\pi - x}{2n + 1} + \text{Cot. } \frac{\pi + x}{2n + 1} - \text{Cot. } \frac{2\pi - x}{2n + 1} \dots + \text{Cot. } \frac{(n\pi + 1)\pi + x}{2n + 1}$.

Denotando igitur $\frac{1}{x} = \frac{1}{m} \text{Cot. } \frac{x}{m}; \frac{1}{\pi-x} = \frac{1}{m} \text{Cot. } \frac{\pi-x}{m};$
 $\frac{1}{\pi+x} = \frac{1}{m} \text{Cot. } \frac{\pi+x}{m}; \frac{1}{2\pi-x} = \frac{1}{m} \text{Cot. } \frac{2\pi-x}{m};$ etc. resp. per $\xi', \xi'',$
 ξ''' etc., aut per $\eta', \eta'', \eta''',$ prout sit $m = 2n$, aut $= 2n+1$, erit
 $\text{Cot. } x = \phi(x, n) = (\xi' - \xi'' + \xi''' \dots - \xi^{2n}) = \psi(x, n)$
 $= (\eta' - \eta'' + \eta''' \dots + \eta^{2n+1}).$ Est autem (§. 6), pro $y < z < \frac{1}{2}\pi$,
 $\frac{1}{y} - \text{Cot. } y > \frac{1}{z} - \text{Cot. } z$, adeoque pro $x < \frac{1}{2}\pi$, $\xi' < \xi'' < \xi''' < \text{etc.}$
et $\eta' < \eta'' < \eta''' < \text{etc.}$; erit ergo $\xi' - \xi'' + \xi''' \dots - \xi^{2n}$ quanti-
tas negativa et $\eta' - \eta'' + \eta''' \dots + \eta^{2n+1}$ quantitas positiva
et $\phi(x, n) < \text{Cot. } x < \psi(x, n)$, unde $\text{Cot. } x = \lim. \phi(x, n) =$
 $\lim. \psi(x, n)$ q. e. d.

§. 19.

Coroll. Cum etiam $\psi(x, n)$ per $\frac{1}{x} - \frac{2x}{\pi^2-x^2} - \frac{2x}{4\pi^2-x^2}$
 $- \frac{2x}{9\pi^2-x^2} \dots - \frac{2x}{n^2\pi^2-x^2}$, $\phi(x, n)$ per $\frac{\pi-2x}{x(\pi-x)} + \frac{\pi-2x}{(\pi+x)(2\pi-x)}$
 $+ \frac{\pi-2x}{(2\pi+x)(3\pi-x)} \dots + \frac{\pi-2x}{((n-1)\pi+x)(n\pi-x)}$, ideoque $\phi(\frac{1}{2}\pi-x, n)$
per $\frac{8x}{\pi^2-4x^2} + \frac{8x}{9\pi^2-4x^2} + \frac{8x}{25\pi^2-4x^2} \dots + \frac{8x}{(2n-1)^2\pi^2-4x^2}$ ex-
hiberi possint, habebitur

$$\text{Cot. } x = \frac{1}{x} - \lim. \left(\frac{2x}{\pi^2-x^2} + \frac{2x}{4\pi^2-x^2} + \frac{2x}{9\pi^2-x^2} \dots + \frac{2x}{n^2\pi^2-x^2} \right)$$

$$\text{et tan. } x = \lim. \left(\frac{8x}{\pi^2-4x^2} + \frac{8x}{9\pi^2-4x^2} + \frac{8x}{25\pi^2-4x^2} \dots + \frac{8x}{n^2\pi^2-4x^2} \right).$$

§. 20.

Coroll. Nullo negotio perspicitur esse $\psi(\frac{1}{2}x, n) = \psi(x, 2n)$

$$\begin{aligned}
&= \frac{1}{x} + \frac{2x}{\pi^2 - x^2} - \frac{2x}{4\pi^2 - x^2} + \frac{2x}{9\pi^2 - x^2} \cdots - \frac{2x}{4n^2\pi^2 - x^2} = \frac{1}{x} + \frac{1}{\pi - x} \\
&- \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} \cdots - \frac{1}{(2n-1)\pi + x} - \frac{1}{2n\pi - x} + \frac{1}{2n\pi + x} \\
&= \frac{\pi}{x(\pi - x)} - \frac{3\pi}{(\pi + x)(2\pi - x)} + \frac{5\pi}{(2\pi + x)(3\pi - x)} \cdots - \frac{(4n-1)\pi}{((2n-1)\pi + x)(2n\pi - x)} \\
&- \frac{1}{2n\pi + x}. \text{ Erit ergo } \lim. \psi\left(\frac{1}{2}x, n\right) = \lim. \psi(x, 2n) = \text{Cot. } \frac{1}{2}x -
\end{aligned}$$

$$\begin{aligned}
&\text{Cot. } x = \text{Cosec. } x = \frac{1}{x} + \lim. \left(\frac{2x}{\pi^2 - x^2} - \frac{2x}{4\pi^2 - x^2} \cdots - \frac{2x}{4n^2\pi^2 - x^2} \right) \text{ aut} = \lim. \\
&\left(\frac{\pi}{x(\pi - x)} - \frac{3\pi}{(\pi + x)(2\pi - x)} + \frac{5\pi}{(2\pi + x)(3\pi - x)} \cdots - \frac{(4n-1)\pi}{((2n-1)\pi + x)(2n\pi - x)} \right), \\
&\text{adeoque etiam } \text{Cosec. } \left(\frac{1}{2}\pi - x \right) \text{ aut } \text{Sec. } x = \\
&\lim. \left(\frac{4\pi}{\pi^2 - 4x^2} - \frac{4 \cdot 3\pi}{9\pi^2 - 4x^2} + \frac{4 \cdot 5\pi}{25\pi^2 - 4x^2} \cdots - \frac{4 \cdot (4n-1)\pi}{(4n-1)^2\pi^2 - 4x^2} \right).
\end{aligned}$$

§. 21.

Theorema. Si ponatur $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots + \frac{1}{n^{2r}} = S^{r,n}$

$$\lim. S^{r,n} = S^r \text{ erit } \text{Cot. } x = \frac{1}{x} - 2 \lim. \left(\frac{S'}{\pi^2} x + \frac{S''}{\pi^4} x^3 + \frac{S'''}{\pi^6} x^5 \cdots + \frac{S^n}{\pi^{2n}} x^{2n-1} \right).$$

Demonstr. Facile e serie $\psi(x, n)$ (§. 19) patet, statuendo

$$\frac{x^{2n}}{x^2} \left(\frac{x}{\pi^2 - x^2} + \frac{x}{4^n(4\pi^2 - x^2)} + \frac{x}{9^n(9\pi^2 - x^2)} \cdots + \frac{x}{n^{2n}(n\pi^2 - x^2)} \right) = \omega_n$$

$$\text{et } (S' - S^{1,n}) \frac{x}{\pi^2} + (S'' - S^{2,n}) \frac{x^3}{\pi^4} + (S''' - S^{3,n}) \frac{x^5}{\pi^6} \cdots + (S^n - S^{n,n}) \frac{x^{2n-1}}{\pi^{2n}} = \zeta,$$

$$\text{fore } \psi(x, n) = \frac{1}{x} - 2 \left(\frac{S'}{\pi^2} x + \frac{S''}{\pi^4} x^3 + \frac{S'''}{\pi^6} x^5 \cdots + \frac{S^n}{\pi^{2n}} x^{2n-1} \right) + 2\zeta - 2\omega_n.$$

Est autem, quia $S' - S^{1,n} > S'' - S^{2,n} > S''' - S^{3,n} > \text{etc.}$, $\zeta < (S' - S^{1,n})$

$$\left(\frac{x}{\pi^2} + \frac{x^3}{\pi^4} \cdots + \frac{x^{2n-1}}{\pi^{2n}} \right) < \frac{x(S' - S^{1,n})}{\pi^2 - x^2} \text{ et } \omega_n < \frac{x^{2n+1}}{\pi^{2n}} \left(1 + \frac{1}{4^n} + \frac{1}{9^n} \cdots + \frac{1}{n^{2n}} \right)$$

$< \frac{x^{2n+1}}{\pi^{2n}} S^n$; utraque ergo quantitas ζ_n , ω_n quant. data minor fieri potest. Hinc $\lim. \psi(x, n)$ aut

$$\text{Cot. } x = \frac{1}{x} - 2 \lim. \left(\frac{S'}{\pi^2} x + \frac{S''}{\pi^4} x^3 \dots + \frac{S^n}{\pi^{2n}} x^{2n-1} \right).$$

§. 22.

Simili modo demonstrari potest denotando

$$\lim. \left(1 + \frac{1}{9^r} + \frac{1}{25^r} \dots + \frac{1}{(2n+1)^r} \right) \text{ per } S_r; \lim. \left(1 - \frac{1}{4^r} + \frac{1}{9^r} - \frac{1}{16^r} \dots - \frac{1}{4n^r} \right)$$

$$\text{per } \mathfrak{S}^r \text{ et } \lim. \left(1 - \frac{1}{3^{2r-1}} + \frac{1}{5^{2r-1}} - \frac{1}{7^{2r-1}} \dots - \frac{1}{(4n-1)^{2r-1}} \right) \text{ per } \mathfrak{S}_r \text{ fore}$$

$$\text{tang. } x = \lim. \left(\frac{2^3 S_1}{\pi^2} x + \frac{2^5 S_2}{\pi^4} x^3 + \frac{2^7 S_3}{\pi^6} x^5 \dots + \frac{2^{2n+1} S_n}{\pi^{2n}} x^{2n-1} \right)$$

$$\text{Cosec. } x = \frac{1}{x} + 2 \lim. \left(\frac{\mathfrak{S}'}{\pi^2} x + \frac{\mathfrak{S}''}{\pi^4} x^3 + \frac{\mathfrak{S}'''}{\pi^6} x^5 \dots + \frac{\mathfrak{S}^n}{\pi^{2n}} x^{2n-1} \right)$$

$$\text{Sec. } x = \lim. \left(\frac{4 \mathfrak{S}_1}{\pi} x + \frac{4^2 \mathfrak{S}_2}{\pi^3} x^2 + \frac{4^3 \mathfrak{S}_3}{\pi^5} x^4 \dots + \frac{4^n \mathfrak{S}_n}{\pi^{2n-1}} x^{2n-2} \right)$$

Simul apparet, quia $\text{tang. } x = \text{Cot. } x - 2 \text{ Cot. } 2x$ et $\text{Cosec. } x =$

$$\text{Cot. } \frac{1}{2} x - \text{Cot. } x \text{ fore, } \lim. \left(\frac{4 S_1}{\pi^2} x + \frac{4^2 S_2}{\pi^4} x^3 \dots + \frac{4^n S_n}{\pi^{2n}} x^{2n-1} \right)$$

$$= \lim. \left(\frac{(4-1) S'}{\pi^2} x + \frac{(4^2-1) S''}{\pi^4} x^3 \dots + \frac{(4^n-1) S^n}{\pi^{2n}} x^{2n-1} \right)$$

$$\text{et } \lim. \left(\frac{\mathfrak{S}'}{\pi^2} x^2 + \frac{\mathfrak{S}''}{\pi^4} x^4 \dots + \frac{\mathfrak{S}^n}{\pi^{2n}} x^{2n} \right) =$$

$$= \lim. \left(\frac{(2-1) S'}{2 \pi^2} x^2 + \frac{(2^3-1) S''}{2^3 \pi^4} x^4 \dots + \frac{(2^{2n-1}-1) S^n}{2^{2n-1} \pi^{2n}} x^{2n} \right)$$

$$\text{unde sequitur } S_1 = \frac{4-1}{4} S', \quad S_2 = \frac{4^2-1}{4^2} S'', \quad S_3 = \frac{4^3-1}{4^3} S''' \text{ etc.}$$

$$\mathfrak{S}' = \frac{2-1}{2} S', \quad \mathfrak{S}'' = \frac{2^3-1}{2^3} S'', \quad \mathfrak{S}''' = \frac{2^5-1}{2^5} S''' \text{ etc.}$$

§. 23.

Theorema. Statuendo $\lim \left(1 + \frac{1}{9^r} + \frac{1}{25^r} \dots + \frac{1}{(2n+1)^{2r}} \right) = \frac{r\mathcal{A}^{(r)}}{4^r} \pi_c^{2r}$ quantitates $\mathcal{A}'$, $\mathcal{A}''$, ... $\mathcal{A}^r$ ita comparatae erunt, ut sit, denotando $\frac{m(m-1)}{1} \dots \frac{(m-r+1)}{r}$ per m_c^r , $\mathcal{A}' = 1$; $\mathcal{A}'' = 3\mathcal{A}' - 1 = 2$; $\mathcal{A}''' = 5_c^2 \mathcal{A}'' - 5_c^4 \mathcal{A}' + 1 = 16$; $\mathcal{A}'''' = 7_c^2 \mathcal{A}''' - 7_c^4 \mathcal{A}'' + 7_c^6 \mathcal{A}' - 1 = 272$; etc.

Demonstr. Quia $S_r = \frac{r\mathcal{A}^{(r)}}{4^r} \pi_c^{2r}$ adeoque $\frac{2^{2r+1}}{\pi_c^{2r}} S_r = \frac{\mathcal{A}^r}{1 \cdot 2 \dots (2r-1)}$ erit (§. 22) $\text{tang. } x = \lim (\mathcal{A}'x + \mathcal{A}''x_c^3 + \mathcal{A}'''x_c^5 \dots + \mathcal{A}^{2n}x_c^{4n-1})$. Est autem, n aucto, $\text{tang. } x = \frac{\lim. (x - x_c^3 + x_c^5 \dots - x_c^{4n-1})}{\lim. (1 - x_c^2 + x_c^4 \dots + x_c^{4n})}$, hinc etiam $\lim (x - x_c^3 \dots - x_c^{4n-1}) = \lim (\mathcal{A}'x + \mathcal{A}''x_c^3 \dots + \mathcal{A}^{2n}x_c^{4n-1}) \times (1 - x_c^2 + x_c^4 \dots + x_c^{4n})$ et, denotando $\mathcal{A}'$; $\mathcal{A}'' = 3_c^2 \mathcal{A}'$; $\mathcal{A}''' = 5_c^2 \mathcal{A}'' + 5_c^4 \mathcal{A}'$; $\mathcal{A}'''' = 7_c^2 \mathcal{A}''' + 7_c^4 \mathcal{A}'' - 7_c^6 \mathcal{A}'$; etc. respective per A' , A'' , A''' , etc., $= \lim (A'x + A''x_c^3 \dots A^{2n}x_c^{4n-1} + B'x_c^{4n+1} \dots + B^{2n}x_c^{4n-1})$. Quales cunque autem sint coefficientes B' , B'' , etc. tamen manifestum est $B'x_c^{4n+1} + B''x_c^{4n+3} \dots + B^{2n}x_c^{4n-1}$, valore ipsius x satis parvo et n aucto, quac. quant. data minorem fieri posse. Erit ergo $\lim. (A'x + A''x_c^3 \dots A^{2n}x_c^{4n-1}) = \lim. (x - x_c^3 + x_c^5 \dots - x_c^{4n-1})$, unde nullo negotio sequitur $A' = 1$, $A'' = -1$, $A''' = 1$, etc. aut $\mathcal{A}' = 1$, $\mathcal{A}'' = 3\mathcal{A}' - 1 = 2$, $\mathcal{A}''' = 5_c^2 \mathcal{A}'' - 5_c^4 \mathcal{A}' + 1 = 16$, etc. q. e. d.

§. 24.

Coroll. Ex (§. 22) sponte sequitur

$$\lim \left(1 + \frac{1}{4^r} + \frac{1}{9^r} + \frac{1}{16^r} \dots + \frac{1}{n^{2r}} \right) = \frac{r \mathcal{A}^r}{(4^r - 1)} \cdot \pi_c^{2r}$$

$$\text{et } \lim. \left(1 - \frac{1}{4^r} + \frac{1}{9^r} - \frac{1}{16^r} \dots + \frac{1}{n^{2r}} \right) = \frac{r(2^{2r-1} - 1) \mathcal{A}^r}{2^{2r-1}(4^r - 1)} \pi_c^{2r}$$

§. 25.

Scholion. E numeris $\mathcal{A}'$, $\mathcal{A}''$, $\mathcal{A}'''$, etc. facile numeri in theoria serierum notatu dignissimi et ab Inventore Jacobo Bernoullio, Bernoulliani dicti, deducuntur. Quos enim per $\mathcal{B}'$, $\mathcal{B}''$, $\mathcal{B}'''$, etc. denotando, erit $\mathcal{B}' = \frac{\mathcal{A}'}{2(4-1)} = \frac{1}{6}$; $\mathcal{B}'' = \frac{2\mathcal{A}''}{2^3(4^2-1)} = \frac{1}{30}$;

$$\mathcal{B}''' = \frac{3\mathcal{A}'''}{2^5(4^3-1)} = \frac{1}{42}; \mathcal{B}'''' = \frac{4\mathcal{A}''''}{2^7(4^4-1)} = \frac{1}{30}; \mathcal{B}' = \frac{5\mathcal{A}'}{2^9(4^5-1)} = \frac{5}{66}; \text{ etc.}$$

Erit etiam $1 - \frac{1}{2} x \cot. \frac{1}{2} x = \mathcal{B}' x_c^2 + \mathcal{B}'' x_c^4 + \mathcal{B}''' x_c^6 + \text{ etc.}$

§. 26.

Theorema. Si numeri $\mathcal{D}'$, $\mathcal{D}''$, $\mathcal{D}'''$, etc. ita comparati sunt, ut sit $\mathcal{D}' = 1$; $\mathcal{D}'' = 4^{1/2} \mathcal{D}' - 1 = 5$; $\mathcal{D}''' = 6^{1/2} \mathcal{D}'' - 6^{1/4} \mathcal{D}' + 1 = 61$; $\mathcal{D}'''' = 8^{1/2} \mathcal{D}''' - 8^{1/4} \mathcal{D}'' + 8^{1/6} \mathcal{D}' - 1 = 1385$; etc. erit

$$\lim. \left(1 - \frac{1}{3^{2r-1}} + \frac{1}{5^{2r-1}} - \frac{1}{7^{2r-1}} \dots + \frac{1}{(4n+1)^{2r-1}} \right) = \frac{r \mathcal{D}^r}{4^r} \pi_c^{2r-1}$$

Demonstr. Sit $\sec x = \lim (1 + \mathcal{D} x_c^2 + \mathcal{D}'' x_c^4 \dots + \mathcal{D}^n x_c^{2n})$, erit, quia $\cos x = \lim. (1 - x_c^2 + x_c^4 \dots + x_c^{2n})$ et $\sec x \cos x = 1$, $\lim. (1 + \mathcal{D} x_c^2 + \mathcal{D}'' x_c^4 \dots + \mathcal{D}^n x_c^{2n}) (1 - x_c^2 + x_c^4 \dots + x_c^{2n}) = 1$; unde prorsus simili modo, quo (§. 23) quantitates $\mathcal{A}'$, $\mathcal{A}''$, etc. determinatae sunt, sequitur, $\mathcal{D}' = 1$, $\mathcal{D}'' = 4^{1/2} \mathcal{D}' - 1$; $\mathcal{D}''' = 6^{1/2} \mathcal{D}'' - 6^{1/4} \mathcal{D}' + 1$, etc.

DISQUISITIO SECUNDA

DE FUNCTIONIBUS LOGARITHMICIS ET TRIGONOMETRICIS IN SERIES EVOLVENDIS.

§. 1.

Theorema. Signo ϕx talem functionem quantitatis mutabilis x denotante, ut pro quibuscunque valoribus ipsarum x , y sit $\phi x + \phi y = \phi(x + y)$, erit $\phi x = x \phi 1$.

Demonstr. Quia $\phi x + \phi y = \phi(x + y)$, facile intelligitur, pro quibuscunque numeris integris positivis m , n fore $m \phi x = \phi mx$, $n \phi y = \phi ny$. Si ergo $mx >$, $=$ aut $< ny$ adeoque $\phi mx >$, $=$ aut $< \phi ny$, erit etiam resp. $m \phi x >$, $=$ aut $< n \phi y$. Itaque, ex definitione proportionis ab Euclide data, $x:y = \phi x:\phi y$ adeoque etiam $x:1 = \phi x:\phi 1$ aut $\phi x = x \phi 1$.

§. 2.

Theorema. Designando per ϕx talem functionem ipsius x , ut omnes ejus valores reales sint positivae et $\phi x \phi y = \phi(x + y)$, erit $\phi x = (\phi 1)^x$.

Demonstr. Quando $\phi x \phi y = \phi(x + y)$, erit etiam $\log \phi x + \log \phi y = \log \phi(x + y)$ adeoque (§. 1) $\log \phi x = x \log \phi 1$ et $\phi x = (\phi 1)^x$.

§. 5.

Theorema. Si functiones ψx , ϕx ita comparatae sunt, ut sit $\psi 0 = 1$, $\phi 0 = 0$, $\psi x \psi y - \phi x \phi y = \psi(x+y)$ et $\phi x \psi y + \psi x \phi y = \phi(x+y)$, erit $\frac{\psi x}{\sqrt{\psi_1^2 + \phi_1^2}^x} = \text{Cos } x \text{ Arc tan } \frac{\phi_1}{\psi_1}$

et $\frac{\phi x}{\sqrt{\psi_1^2 + \phi_1^2}^x} = \text{Sin } x \text{ Arc tan } \frac{\phi_1}{\psi_1}$.

Demonstr. Ex duabus aequationibus

$$\psi x \psi y - \phi x \phi y = \psi(x+y)$$

$$\phi x \psi y + \psi x \phi y = \phi(x+y) \quad \text{sequitur}$$

$$(\psi x^2 + \phi x^2)(\psi y^2 + \phi y^2) = \psi(x+y)^2 + \phi(x+y)^2 \quad \text{adeoque}$$

$$\text{erit (§. 2)} \quad \phi x^2 + \psi x^2 = (\psi_1^2 + \phi_1^2)^x \quad \text{et, denotando } \frac{\psi x}{\sqrt{\psi_1^2 + \phi_1^2}^x}$$

$$\text{per } \psi' x, \quad \frac{\psi x}{\sqrt{\psi_1^2 + \phi_1^2}^x} \quad \text{per } \phi' x, \quad \psi' x^2 + \phi' x^2 = 1. \quad \text{Jam ponamus}$$

$$\psi' x = \text{Cos } \xi, \quad \psi' y = \eta, \quad \text{erit } \phi' x = \text{Sin } \xi, \quad \phi' y = \text{Sin } \eta \quad \text{et quia}$$

$$\psi' x \psi' y - \phi' x \phi' y = \psi'(x+y)$$

$$\phi' x \psi' y + \psi' x \phi' y = \phi'(x+y)$$

$$\text{nec non } \text{Cos } \xi \text{ Cos } \eta - \text{Sin } \xi \text{ Sin } \eta = \text{Cos } (\xi + \eta)$$

$$\text{Sin } \eta \text{ Cos } \xi + \text{Cos } \xi \text{ Sin } \eta = \text{Sin } (\xi + \eta)$$

$$\text{erit } \psi'(x+y) = \text{Cos } (\xi + \eta), \quad \phi'(x+y) = \text{Sin } (\xi + \eta) \quad \text{et}$$

$$\frac{\phi'(x+y)}{\psi'(x+y)} = \tan (\xi + \eta). \quad \text{Unde sequitur}$$

$$\text{Arc tan } \frac{\phi x}{\psi x} + \text{Arc tan } \frac{\phi y}{\psi y} = \text{Arc tan } \frac{\phi(x+y)}{\psi(x+y)} \quad \text{adeoque (§. 1)}$$

$$\text{Arc tan } \frac{\phi x}{\psi x} = x \text{ Arc tan } \frac{\phi_1}{\psi_1} \quad \text{et quia } \text{Arc tan } \frac{\phi x}{\psi x} = \text{Arc Cos } \frac{\psi x}{\sqrt{\psi_1^2 + \phi_1^2}^x}$$

$$= \text{Arc Sin } \frac{\phi x}{\sqrt{\psi_1^2 + \phi_1^2 x}}, \text{ erit etiam } \frac{\psi x}{\sqrt{\psi_1^2 + \phi_1^2 x}} = \text{Cos } x \text{ Arc tan } \frac{\phi_1}{\psi_1}$$

$$\text{et } \frac{\phi x}{\sqrt{\psi_1^2 + \phi_1^2 x}} = \text{Sin } x \text{ Arc tan } \frac{\phi_1}{\psi_1} \text{ q. e. d.}$$

§. 4.

Theorema. Signo $x^{(r)}$ talem functionem cujuscunque quantitatis x et numeri positivi integri r denotante, quae, pro $x = 0$, fit $= 0$ et ita comparata est, ut sit

$$x^{(n)} + y^{(1)} x^{(n-1)} + y^{(2)} x^{(n-2)} \dots + y^{(n-2)} x^{(2)} + y^{(n-1)} x^{(1)} + y^{(n)} = (x + y)^{(n)}$$

statuendoque

$$1 + x^{(1)} w + x^{(2)} w^2 + x^{(3)} w^3 + x^{(4)} w^4 + \text{etc.} = f x$$

$$1 - x^{(2)} w^2 + x^{(4)} w^4 - x^{(6)} w^6 + x^{(8)} w^8 - \text{etc.} = \psi x$$

$$\text{et } x^{(1)} w - x^{(3)} w^3 + x^{(5)} w^5 - x^{(7)} w^7 + x^{(9)} w^9 - \text{etc.} = \phi x$$

$$\text{habebitur } 1) f x = (f_1)^x; \quad 2) \frac{\psi x}{\sqrt{\psi_1^2 + \phi_1^2 x}} = \text{Cos } x \text{ Arc tan } \frac{\phi_1}{\psi_1}$$

$$\text{et } 3) \frac{\phi x}{\sqrt{\psi_1^2 + \phi_1^2 x}} = \text{Sin } x \text{ Arc tan } \frac{\phi_1}{\psi_1}.$$

Demonstr. Multiplicationibus rite institutis et resp. $(x + y)^{(1)}$, $(x + y)^{(2)}$, $(x + y)^{(3)}$, etc. loco $x + y$, $x^{(2)} + y x + y^{(2)}$, $x^{(3)} + y^{(1)} x^{(2)} + y^{(2)} x^{(1)} + y^{(3)}$, etc. substitutis nanciscemur
 $f x f y = 1 + (x + y)^{(1)} w + (x + y)^{(2)} w^2 + (x + y)^{(3)} w^3 + \text{etc.}$
 $\psi x \psi y - \phi x \phi y = 1 - (x + y)^{(2)} w^2 + (x + y)^{(4)} w^4 - (x + y)^{(6)} w^6 + \text{etc.}$
 $\phi x \psi y + \psi x \phi y = (x + y)^{(1)} w - (x + y)^{(3)} w^3 + (x + y)^{(5)} w^5 - (x + y)^{(7)} w^7 + \text{etc.}$
aut $f x f y = f(x + y)$; $\psi x \psi y - \phi x \phi y = \psi(x + y)$ et $\phi x \psi y$

$\psi x \phi y = \phi(x + y)$ adeoque erit (§. 2) 1) $fx = (f1)^x$ et,
quia $\psi 0 = 1$, $\phi 0 = 0$, (§. 3) 2) $\frac{\phi x}{\sqrt{\psi 1^2 + \phi 1^2}} = \text{Cos Arc tan } \frac{\phi 1}{\psi 1}$
et 3) $\frac{\phi x}{\sqrt{\psi 1^2 + \phi 1^2}} = \text{Sin Arc tan } \frac{\phi 1}{\psi 1}$ q. e. d.

§. 5.

Theorema. Denotando $\frac{z^r}{1.2.3\dots r}$ per z_c^r , erit

$$x_c^n + x_c^{n-1}y + x_c^{n-2}y^2 + \dots + x_c^2y^{n-2} + xy^{n-1} + y_c^n = (x + y)_c^n.$$

Demonstr. Est $\frac{x+y}{n+1} x_c^{n-r}y_c^r = \frac{n-r+1}{n+1} x_c^{n-r+1}y_c^r + \frac{r+1}{n+1} x_c^{n-r}y_c^{r+1}$

adeoque erit, statuendo $x_c^n + x_c^{n-1}y + \dots + xy_c^{n-1} + y_c^n = \phi_n$,

$$\begin{aligned} \frac{x+y}{n+1} \phi_n &= x_c^{n+1} + \frac{n}{n+1} x_c^n y + \frac{n-1}{n+1} x_c^{n-1}y^2 + \dots + \frac{1}{n+1} x y_c^n \\ &\quad + \frac{1}{n+1} x_c^n y + \frac{2}{n+1} x_c^{n-1}y^2 + \dots + \frac{n}{n+1} x y_c^n + y_c^{n+1} \\ &= x_c^{n+1} + x_c^n y + x_c^{n-1}y^2 + \dots + x y_c^n + y_c^{n+1} = \phi(n+1). \end{aligned}$$

Cum autem sit $x + y = \phi_1$, erit $(x + y)_c^2 = \frac{x+y}{2} \phi_1 = \phi_2$; $(x + y)_c^3 = \frac{x+y}{3} \phi_2 = \phi_3$; etc. et generaliter $(x + y)_c^n = \phi_n$ q. e. d.

§. 6.

Coroll. Designando $z(z-1)(z-2)\dots(z-r+1)$ per z^r et $\frac{z^r}{1.2\dots r}$ per z_c^r , habebitur, quia $\frac{n^n}{(n-r)^{n-r}r^r} = n_c^r = n_c^{n-r}$

adeoque $n^n x_c^{n-r}y_c^r = n_c^r x^{n-r}y^r = n_c^{n-r} x^{n-r}y^n$, pro quocunque valore positivo integro ipsius n

$$(x + y)^n = x^n + n x^{n-1}y + n_c^2 x^{n-2}y^2 + \dots + n x y^{n-1} + y^n.$$

§. 7.

Theorema. Significatione notationis z_c^r (§. prec.) salva erit $(x+y)_c^n = x_c^n + x_c^{n-1}y + x_c^{n-2}y_c^2 \dots + xy_c^{n-1} + y_c^n$.

Demonstr. Cum sit $\frac{x+y-n}{n+1} x_c^{n-r} y_c^r = \frac{x-n+1}{n+1} x_c^{n-r} y_c^r$
 $+ x_c^{n-r} y_c^r \frac{y-r}{n+1} = \frac{n-r+1}{n+1} x_c^{n-r+1} y_c^r + \frac{r+1}{n+1} x_c^{n-r} y_c^{r+1}$ erit,
 denotando $x_c^n + x_c^{n-1}y + x_c^{n-2}y_c^2 \dots + xy_c^{n-1} + y_c^n$ per ϕ_n ,
 $\frac{x+y-n}{n+1} \phi_n = x_c^{n+1} + \frac{n}{n+1} x_c^n y + \frac{n-1}{n+1} x_c^{n-1} y_c^2 \dots + \frac{1}{n+1} xy_c^n$
 $+ \frac{1}{n+1} x_c^n y + \frac{2}{n+1} x_c^{n-1} y_c^2 \dots + \frac{n}{n+1} xy_c^n + y_c^{n+1}$
 $= x_c^{n+1} + x_c^n y + x_c^{n-1} y_c^2 \dots + xy_c^n + y_c^{n+1} = \phi_{n+1}$
 Ergo $x+y = \phi_1$; $(x+y)_c^2 = \frac{x+y-1}{2} \cdot (x+y) = \frac{x+y-1}{2} \phi_1 = \phi_2$;
 $(x+y)_c^3 = \frac{x+y-2}{3} \cdot (x+y)_c^2 = \frac{x+y-2}{3} \phi_2 = \phi_3$; etc. et generaliter
 $(x+y)_c^n = \frac{x+y-n+1}{n} \cdot (x+y)_{c^{n-1}} = \frac{x+y-n+1}{n} \phi_{n-1} = \phi_n$. q. e. d.

§. 8.

Theorema. $\frac{x}{n} (x+n\theta-1)_c^{n-1} + y \cdot \frac{x}{n-1} (x+(n-1)\theta-1)_c^{n-2}$
 $+ \frac{y}{2} (y+2\theta-1) \cdot \frac{x}{n-2} (x+(n-2)\theta-1)_c^{n-3} + \frac{y}{3} (y+3\theta-1)_c^2 \times$
 $\frac{x}{n-3} (x+(n-3)\theta-1)_c^{n-4} \dots + \frac{y}{n-1} (y+(n-1)\theta-1)_c^{n-2} x$
 $+ \frac{y}{n} (y+n\theta-1)_c^{n-1} = \frac{x+y}{n} (x+y+n\theta-1)_c^{n-1}$.

Demonstr. Est $\frac{x}{n} (x + n\theta - 1)_c^{m-1} = \frac{x + n\theta}{n} (x + n\theta - 1)_c^{m-2} -$

$$\theta(x+n\theta-1)_c^{n-1} = (x+n\theta)_c^n - \theta(x+n\theta-1)_c^{n-1} \text{ adeoque etiam}$$
$$\frac{x+y}{n} (x+y+n\theta-1)_c^{n-1} = (x+y+n\theta)_c^n - \theta(x+y+n\theta-1)_c^{n-1}$$

Cum autem (§. 7) sit

$$(x+y+n\theta)_c^n = (y+n\theta)_c^n + x(y+n\theta)_c^{n-1} \dots + x_c^{n-1}(y+n\theta) + x_c^n$$

$$\theta(x+y+n\theta-1)_c^{n-1} = \theta(y+n\theta-1)_c^{n-1} + \theta x(y+n\theta-1)_c^{n-2} \dots + \theta x_c^{n-1}, \text{ erit}$$

$$\frac{x+y}{n} (x+y+n\theta-1)_c^{n-1} = \frac{y}{n} (y+n\theta-1)_c^{n-1} + x \frac{y+\theta}{n-1} (y+n\theta-1)_c^{n-2}$$

$$+ x_c^2 \frac{y+2\theta}{n-2} (y+n\theta-1)_c^{n-3} \dots + x_c^n \text{ aut, denotando brevitatis gratia}$$

$$\frac{z}{r} (z + r\theta - 1)_c^{r-1} \text{ per } z_c^{(r)}, (x + y)_c^{(n)} = y_c^{(n)} + x(y + \theta)_c^{(n-1)} + x^2(y + 2\theta)_c^{(n-2)}$$

$+ x_c^3 (y + 5\theta)_c^{(n-3)} \dots + x_c^n$. Quodsi jam statuimus theorema nostrum

pro omnibus valoribus ipsius n inde ab $n = 1$ usque ad $n-1$ verum

esse, erit $(x + y)_c^{(n)} = y^{(n)} +$

$$\begin{array}{c|c|c} x y_e^{(n-1)} + x \theta & y_c^{(n-1)} + x \theta_c^{(2)} & y_c^{(n-3)} \dots \dots + x \theta_c^{(n-1)} \\ + x_c^{i2} & + x_c^{i2} 2\theta & \dots \dots + x_c^{i2} (2\theta)_c^{(n-2)} \\ & + x_c^{i3} & \dots \dots + x_c^{i3} (3\theta)_c^{(n-3)} \\ & + x_c^{i4} & \dots \dots + x_c^{i4} (4\theta)_c^{(n-4)} \\ & & \vdots \\ & & \dots \dots + x_c^{in} \end{array}$$

Quia autem, existente $m < r$, $(m\theta)_c^{(r-m)} = \frac{m\theta}{r-m} (m\theta + (r-m)\theta - 1)_c^{r-m-1}$

$$= \frac{m\theta}{r-m} (r\theta - 1)^{r-m-1} = \frac{m}{r} (r\theta)^{r-m}, \text{ erit}$$

$$\begin{aligned}
& x \theta_c^{(r-1)} + x_c^2 (2\theta_c)^{(r-2)} + x_c^3 (3\theta_c)^{(r-3)} + x_c^4 (4\theta_c)^{(r-4)} \dots + x_c^r \\
&= \frac{x}{r} (r\theta_c)^{(r-1)} + \frac{2}{r} x_c^2 (r\theta_c)^{(r-2)} + \frac{3}{r} x_c^3 (r\theta_c)^{(r-3)} + \frac{4}{r} x_c^4 (r\theta_c)^{(r-4)} \dots + r x_c^r \\
&= \frac{x}{r} \left((r\theta_c)^{(r-1)} + (x-1)(r\theta_c)^{(r-2)} + (x-1)^2 (r\theta_c)^{(r-3)} + (x-2)^2 (r\theta_c)^{(r-4)} \dots + (x-1)^{(r-1)} \right) \\
&= (\S. 7) \frac{x}{r} (x + r\theta - 1)_c^{r-1} = x_c^{(r)}, \text{ adeoque}
\end{aligned}$$

$$(x+y)_c^{(n)} = y_c^{(n)} + x y_c^{(n-1)} + x_c^{(2)} y_c^{(n-2)} \dots + x_c^{(n-1)} y + x_c^{(n)}$$

Cum igitur theorema verum sit, pro $n=1$, erit etiam verum pro $n=2$, adeoque verum pro $n=3$ et sic porro pro omnibus sequentibus valoribus ipsius n .

§. 9.

Coroll. Aequationem $(x+y)_c^{(n)} = y_c^{(n)} + x y_c^{(n-1)} \dots + x_c^{(n-1)} y + x_c^{(n)}$ adhuc locum habere, statuendo

$$\frac{z}{r} \frac{(z+r\beta-\alpha)(z+r\beta-2\alpha)(z+r\beta-3\alpha) \dots (z+r\beta-(r-1)\alpha)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (r-1)} = z_c^{(r)}$$

nullo negotio ex theor. §. prec. derivari potest.

§. 10.

Theorema. Denotando per la logarithmum naturalem, cujus basis $e = 1 + 1 + 1_c^2 + 1_c^3 + 1_c^4 + \text{etc.}$

erit $a^x = 1 + x la + x_c^2 (la)^2 + x_c^3 (la)^3 + x_c^4 (la)^4 + \text{etc.}$

Demonstr. Statuendo $1 + xw + x_c^2 w^2 + x_c^3 w^3 + \text{etc.} = fx$, erit $1 + w + w_c^2 + w_c^3 + \text{etc.} = f1$ et proprietatis coefficientium $x, x_c^2, x_c^3, \text{etc.}$ causa (§. 5) debet esse (§. 4) $(f1)^x = fx$ aut

$$(1 + w + w_c^2 + w_c^3 + \text{etc.})^x = 1 + xw + x_c^2 w^2 + x_c^3 w^3 + \text{etc.}$$

Unde etiam $(1 + 1 + 1_c^2 + 1_c^3 + \text{etc.})^x$ aut

$$e^x = 1 + x + x_c^2 + x_c^3 + x_c^4 + \text{etc.}$$

$$\text{et } e^{wx} = 1 + xw + x_c^2 w^2 + x_c^3 w^3 + x_c^4 w^4 + \text{etc.}$$

ergo, existente $e^w = a$, adeoque $w = la$

$$a^x = 1 + x la + x_c^2 (la)_c^2 + x_c^3 (la)_c^3 + x_c^4 (la)_c^4 + \text{etc. q. e. d.}$$

§. 11.

Theorema. Pro quocunque valore rationali aut irrationali ipsius x existenteque $w < 1$ erit

$$(1 + w)^x = 1 + xw + x_c^2 w^2 + x_c^3 w^3 + x_c^4 w^4 + \text{etc.}$$

Demonstr. Denotando $1 + xw + x_c^2 w^2 + \text{etc.}$ per fx , erit, quia $1_c^2, 1_c^3, \text{etc.} = 0$, $1 + w = f_1$. Proprietatis autem coefficientium $x, x_c^2, x_c^3 \text{ etc.}$ (§. 7) demonstratae causa est (§. 4) $(f_1)^x = fx$ aut $(1 + w)^x = 1 + xw + x_c^2 w^2 + x_c^3 w^3 + \text{etc.}$

§. 12.

$$\text{Theorema. } \log.\text{nat.}(1 + w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \text{etc.}$$

Demonstr. Est (§. 10).

$$(1 + w)^x = 1 + x l(1 + w) + x_c^2 (l(1 + w))^2 + x_c^3 (l(1 + w))^3 + \text{etc.}$$

unde facile apparet, fore, x decrescente, $\lim \frac{(1 + w)^x - 1}{x} = \log(1 + w)$.

Quia autem (§. 11)

$$\frac{(1+w)^x - 1}{x} = w - (1-x)\frac{w^2}{2} + (1-x)(1-\frac{x}{2})\frac{w^3}{3} - (1-x)(1-\frac{x}{2})(1-\frac{x}{3})\frac{w^4}{4} + \text{etc.}$$

nullo negotio patet, etiam oportere esse

$$\lim \frac{(1+w)^x - 1}{x} = w - \frac{w^2}{2} + \frac{w^3}{4} - \frac{w^4}{4} + \text{etc.}$$

$$\text{Hinc } \log(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \text{etc.}$$

$$\text{Unde statim sequitur } \log(1-w) = -w - \frac{w^2}{2} - \frac{w^3}{3} - \frac{w^4}{4} - \text{etc.}$$

$$\text{et } \log \frac{1+w}{1-w} = 2 \left(w + \frac{w^3}{3} + \frac{w^5}{5} + \frac{w^7}{7} + \text{etc.} \right)$$

§. 13.

$$\text{Theorema. } \left(1 + w + \theta w^2 + \frac{\theta}{2} (3\theta - 1) w^3 + \frac{\theta}{3} (4\theta - 1)^2 w^4 + \text{etc.} \right)^x \\ = 1 + xw + \frac{x}{2} (x + 2\theta - 1) w^2 + \frac{x}{3} (x + 3\theta - 1)^2 w^3 + \text{etc.}$$

Demonstr. Statuendo $1 + xw + \frac{x}{2} (x + \theta - 1) w^2 + \frac{x}{3} (x + 3\theta - 1)^2 w^3 + \text{etc.} = f_x$ erit $1 + w + \theta w^2 + \frac{\theta}{2} (3\theta - 1) w^3 + \text{etc.} = f_1$ et propter proprietatem coefficientium functionis f_x (§. 8) demonstratam oportet esse (§. 4) $f_x = (f_1)^x$ q. e. d.

§. 14.

Coroll. Substituendo θ loco ipsius x habebitur

$$f_\theta = 1 + \theta w + \frac{\theta}{2} (3\theta - 1) w^2 + \frac{\theta}{3} (4\theta - 1)^2 w^3 + \frac{\theta}{4} (5\theta - 1)^3 w^4 + \text{etc.} \\ = \frac{f_1 - 1}{w} \text{ adeoque, statuendo } f_1 = z, \quad wz^\theta - z + 1 = 0. \quad \text{Data}$$

igitur aequatione $wz^\theta - z + 1 = 0$ invenitur potestas radicis

$$z^x = 1 + xw + \frac{x}{2}(x + 2\theta - 1)w^2 + \frac{x}{3}(x + 3\theta - 1)^2 w^3 + \text{etc.}$$

§. 15.

Theorema. Denotante x arcum quemcunque circuli, cujus radius = 1, erit $\text{Cos } x = 1 - x_c^2 + x_c^4 - x_c^6 + x_c^8 - \text{etc.}$

$$\text{Sin } x = x - x_c^3 + x_c^5 - x_c^7 + x_c^9 - \text{etc.}$$

Demonstr. Statuendo $1 - x_c^2 w^2 + x_c^4 w^4 - x_c^6 w^6 + \text{etc.} = \psi x$ et $xw - x_c^3 w^3 + x_c^5 w^5 - x_c^7 w^7 + \text{etc.} = \phi x$ erit, quia (§. 5) coefficients x_c^r proprietate functionum (§. 4) gaudent

$$\frac{\psi x}{\sqrt{\psi_1^2 + \phi_1^2 x}} = \text{Cos } x \text{ Arc tan } \frac{\phi_1}{\psi_1} \text{ et } \frac{\phi x}{\sqrt{\psi_1^2 + \phi_1^2 x}} = \text{Sin } x \text{ Arc tan } \frac{\phi_1}{\psi_1}$$

Facile autem perspicitur, esse $\psi(-x) = \psi x$, $\phi(-x) = -\phi x$ adeoque, quia $\psi x \psi y - \phi x \phi y = \psi(x+y)$, $\psi x \psi(-x) - \phi x \phi(-x)$ aut $\psi x^2 + \phi x^2 = \psi(x-x) = \psi 0 = 1$. Itaque posito

$$\text{Arc tan } \frac{\phi_1}{\psi_1} = \zeta, \text{ erit}$$

$$1 - x_c^2 w^2 + x_c^4 w^4 - x_c^6 w^6 + x_c^8 w^8 - \text{etc.} = \text{Cos } x \zeta$$

$$xw - x_c^3 w^3 + x_c^5 w^5 - x_c^7 w^7 + x_c^9 w^9 - \text{etc.} = \text{Sin } x \zeta$$

$$\text{Unde sequitur } \text{Cos } x = 1 - x_c^2 \left(\frac{w}{\zeta}\right)^2 + x_c^4 \left(\frac{w}{\zeta}\right)^4 - x_c^6 \left(\frac{w}{\zeta}\right)^6 + \text{etc.}$$

$$\text{Sin } x = x \frac{w}{\zeta} - x_c^3 \left(\frac{w}{\zeta}\right)^3 + x_c^5 \left(\frac{w}{\zeta}\right)^5 - x_c^7 \left(\frac{w}{\zeta}\right)^7 + \text{etc.}$$

Cum autem $\frac{\text{Sin } x}{x} < 1 < \frac{\tan x}{x}$ et (Disq. I, §. 4), $\frac{\text{Sin } x}{x}$, x diminuto

semper crescat dum $\frac{\tan x}{x}$ decrescat et differentia $\frac{\tan x}{x} - \frac{\text{Sin } x}{x}$

$$= \tan x \frac{1 - \cos x}{x} < 4 \sin^2 \frac{x}{2} \text{ quac. quant. data minor fieri possit,}$$

erit $\lim \frac{\sin x}{x} = \lim \frac{\tan x}{x} = 1$. Ex serie autem pro $\sin x$ modo in-

venta simul apparet esse $\lim \frac{\sin x}{x} = \frac{w}{z}$, adeoque $\frac{w}{z} = 1$. Ergo

$$\cos x = 1 - x^2 - x^4 - \text{etc.}, \quad \sin x = x - x^3 + x^5 - \text{etc. q. e. d.}$$

§. 16.

Theorema. Pro quibuscunque valoribus ipsarum x et θ , existente $\theta < 1$, erit

$$\frac{\cos x \operatorname{Arc} \tan \theta}{(\cos \operatorname{Arc} \tan \theta)^x} = 1 - x^2 \theta^2 + x^4 \theta^4 - x^6 \theta^6 + \text{etc.}$$

$$\frac{\sin x \operatorname{Arc} \tan \theta}{(\cos \operatorname{Arc} \tan \theta)^x} = x\theta - x^3 \theta^3 + x^5 \theta^5 - x^7 \theta^7 + \text{etc.}$$

Demonstr. Si $1 - x^2 \theta^2 + x^4 \theta^4 - x^6 \theta^6 + \text{etc.} = \psi x$ et $x\theta - x^3 \theta^3 + x^5 \theta^5 - x^7 \theta^7 + \text{etc.} = \phi x$, erit proprietatis coefficientium x, x^2, x^3 etc. causa (§. 4, 6)

$$\frac{\psi x}{\sqrt{\psi_1^2 + \phi_1^2}} = \cos x \operatorname{Arc} \tan \frac{\phi_1}{\psi_1}; \quad \frac{\phi x}{\sqrt{\psi_1^2 + \phi_1^2}} = \sin x \operatorname{Arc} \tan \frac{\phi_1}{\psi_1}$$

Est autem, quia $1^2, 1^3, 1^4$, etc. = 0, $\psi_1 = 1$ et $\phi_1 = \theta$ adeoque

$$\sqrt{\psi_1^2 + \phi_1^2} = \sqrt{1 + \theta^2} = \sec \operatorname{Arc} \tan \theta = \frac{1}{\cos \operatorname{Arc} \tan \theta} \text{ et } \frac{\phi_1}{\psi_1} = \theta. \text{ Ergo}$$

$$\psi x = \frac{\cos x \operatorname{Arc} \tan \theta}{(\cos \operatorname{Arc} \tan \theta)^x} \text{ et } \phi x = \frac{\sin x \operatorname{Arc} \tan \theta}{(\cos \operatorname{Arc} \tan \theta)^x} \text{ q. e. d.}$$

§. 17.

$$\textbf{Theorema.} \quad \operatorname{Arc} \tan \theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} + \text{etc.}$$

Demonstr. Ex §. prec. facile apparet, esse

$$\tan x \operatorname{Arc} \tan \theta = \frac{x\theta - x^3\theta^3 + x^5\theta^5 - x^7\theta^7 + \text{etc.}}{1 - x^2\theta^2 + x^4\theta^4 - x^6\theta^6 + \text{etc.}} \text{ adeoque}$$

$$\frac{\tan x \operatorname{Arc} \tan \theta}{x} = \frac{\theta - (1-x)\left(1 - \frac{x}{2}\right)\frac{\theta^3}{3} + (1-x)\left(1 - \frac{x}{2}\right) \dots \left(1 - \frac{x}{4}\right)\frac{\theta^5}{5}}{1 - x^2\theta^2 + x^4\theta^4 - x^6\theta^6 + \text{etc.}}$$

$$\text{cujus limitem, } x \text{ decrescente, esse} = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} + \text{etc.}$$

manifestum est. Quia autem (§. 15) $\lim \frac{\tan x}{x} = 1$ adeoque

$$\lim \frac{\tan x \operatorname{Arc} \tan \theta}{x} = \operatorname{Arc} \tan \theta, \text{ erit}$$

$$\operatorname{Arc} \tan \theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} + \frac{\theta^9}{9} - \text{etc.}$$

§. 18.

Theorema. Pro quocunque valore quantitatis x est
 $\cos x \operatorname{Arc} \sin w = 1 - \frac{x^2}{2}w^2 + \frac{x^2(x^2-4)}{1 \cdot 2 \cdot 3 \cdot 4}w^4 - \frac{x^2(x^2-4)(x^2-16)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}w^6 + \text{etc.}$

$$\sin x \operatorname{Arc} \sin w = xw - \frac{x(x^2-1)}{1 \cdot 2 \cdot 3}w^3 + \frac{x(x^2-1)(x^2-9)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 4}w^5 - \text{etc.}$$

Demonstr. Designando $1 - \frac{x^2}{2}w^2 + \frac{x^2(x^2-4)}{1 \cdot 2 \cdot 3 \cdot 4}w^4 - \text{etc.}$ per

$$\psi x \text{ et } xw - \frac{x(x^2-1)}{1 \cdot 2 \cdot 3}w^3 + \frac{x(x^2-1)(x^2-9)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}w^5 - \text{etc. per } \phi x \text{ erit, quia,}$$

coefficientes $x, \frac{x^2}{2}, \frac{x(x^2-1)}{1 \cdot 2 \cdot 3}, \frac{x^2(x^2-4)}{1 \cdot 2 \cdot 3 \cdot 4}, \text{ etc. (in formula generali}$

$x^{(n)}$ (§. 9), si $\alpha = 2 \beta = 1$, continentur, propter (§. 9. 4)

$$\frac{\phi x}{\psi_1 + \phi_1^{2x}} = \cos x \operatorname{Arc} \tan \frac{\phi_1}{\psi_1}, \quad \frac{\phi x}{\psi_1 + \phi_1^{2x}} = \sin x \operatorname{Arc} \tan \frac{\phi_1}{\psi_1}$$

Est autem $\psi_1 = 1 - \frac{1}{2}w^2 + \frac{1 \cdot 1}{2 \cdot 4}w^4 - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}w^6 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}w^8 - \text{etc.}$

$= (\S. 11) (1 - w^2)^{\frac{1}{2}}$ et $\phi_1 = w$ adeoque $\sqrt{\psi_1^2 + \phi_1^2} = 1$

$\frac{\phi_1}{\psi_1} = \frac{w}{\sqrt{1 - w^2}}$ et $\text{Arc tan } \frac{\phi_1}{\psi_1} = \text{Arc tan } \frac{w}{\sqrt{1 - w^2}} = \text{Arc Sin } w$

Hinc $\psi x = \text{Cos } x \text{ Arc Sin } w$ et $\phi x = \text{Sin } x \text{ Arc Sin } w$.

§. 19.

Coroll. Cum $\frac{\text{sit Sin } x \text{ Arc Sin } w}{w} = w + (1 - x^2)^{\frac{1}{2}} \frac{w^3}{3}$

$+ (1 - x^2) \left(1 - \frac{x^2}{9}\right) \frac{1 \cdot 3}{2 \cdot 4} \frac{w^5}{5} + (1 - x^2) \left(1 - \frac{x^2}{9}\right) \left(1 - \frac{x^2}{25}\right) \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{w^7}{7} + \text{etc.}$

adeoque $\lim \frac{\text{Sin } x \text{ Arc Sin } w}{x} = w + \frac{1}{2} \frac{w^3}{5} + \frac{1 \cdot 3}{2 \cdot 4} \frac{w^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{w^7}{7} + \text{etc.}$

nec non (§. 15) $\lim \frac{x \text{ Arc Sin } w}{x} = \text{Arc Sin } w$ erit

$\text{Arc Sin } w = w + \frac{1}{2} \frac{w^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{w^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{w^7}{7} + \text{etc.}$

§. 20

Theorema generale (§. 3) maxima etiam cum utilitate ad demonstrationem parallelogrammi virium Illustr. la Place (Mec. cel. tom 1. §. 1) simpliciore reddendam adhiberi potest.

I. Sint P, Q duae vires in punctum materiale M secundum directiones inter se perpendiculares simul agentes et R vis ex illis composita, cujus directio MC cum directione MA angulum x formet, manifestum est, et R et x functiones virium P, Q et vicissim utramque virium P, Q functionem quantitatum R, x oportere esse. Simul autem apparet, quia ratione P: Q immutata, rationes P: R; Q: R

et angulus x non mutantur, statui posse $P = R \psi x$, $Q = R \phi x$. Jam de indole functionum ψx , ϕx sequentia sunt notanda: Existente $Q = 0$, erit $R = P$, $x = 0$ adeoque $\psi 0 = 1$, $\phi 0 = 0$ et, angulo recto pro unitate sumto, erit, si $P = 0$, $R = Q$, $x = 1$ ergo $\psi 1 = 0$, $\phi 1 = 1$.

II. Utraque virium P , Q in duas alias P' , P'' ; Q' , Q'' decomposita, ita ut P' , Q' secundum MC et P'' , Q'' secundum directionem MD in MC perpendiculararem agant, erit $P' = P \psi x = \frac{P^2}{R}$
 $Q' = Q \phi x = \frac{Q^2}{R}$; $P'' = \frac{PQ}{R}$, $Q'' = Q \psi x = \frac{PQ}{R}$. Cum vires P'' , Q'' inter se sint aequales et oppositae, oportet ut sit $P' + Q' = R$ adeoque $R^2 = P^2 + Q^2$.

III. Quodsi, vi P immutata, Q vi q augeatur, vis composita R nec non angulus x crescet. Denotando illam per R' et hunc per $x + y$ erit $P = R' \psi(x + y)$, $Q + q = R' \phi(x + y)$, atque, incremento q in duas alias vires q' , q'' resp. secundum MC , MD agentes, decomposito, habebitur $q'' = q \psi x$, $q' = q \phi x$, $R + q' = R' \psi y$, $q'' = R' \phi y$. Unde

$$\psi x \psi y - \phi x \phi y = \frac{R q'}{R' q} = \frac{R \psi x}{R'} = \frac{P}{R'} = \psi(x + y)$$

$$\phi x \psi y + \psi x \phi y = \frac{R q' + q'^2 + q''^2}{q R'} = \frac{R \phi x}{R'} + \frac{q^2}{q R'} = \frac{Q + q}{R'}$$

$$= \phi(x + y). \quad \text{Itaque, cum sit } \psi 1^2 + \phi 1^2 = 1, \text{ Arc tan } \frac{\phi 1}{\psi 1}$$

aut $\text{Arc Sin } \phi 1 = \frac{\pi}{2}$ oportet esse $\psi x = \text{Cos } x$, $\phi x = \text{Sin } x$, ita

ut, viribus P , Q per latera rectanguli exhibitis, vis composita R quoad intensitatem et directionem per diagonalem representetur.

DISQUISITIO TERTIA

DE THEOREMATE POLYNOMIALI, DE REVERSIONE SERIERUM ET DE SUMMIS POTESTATUM RADICUM AEQATIONUM CUM COEFFICIENTIBUS EARUM COMPARANDIS.

§. 1.

Definitio. Designabimus in hac disquisitione potestatem divisam $\frac{a^n}{1 \cdot 2 \cdot 3 \dots n}$, ut antea, per a_c^n et per $[m, r]$ summam omnium productorum, quae ex quantitatibus seriei a, a', a'', a''', a'''' etc. earumque potestatibus divisis ita formari possunt, ut cujusque talis producti dimensio sit $= m$ et summa indicum litteris a superne affixarum $= r$. E. g. $[7, 5] = a_c^6 a^v + a_c^5 (a' a'''' + a'' a''') + a_c^4 (a'^2 a''' + a' a''^2) + a_c^3 a'^3 a'' + a_c^2 a'^5$. Per signa $[m, r]', [m, r]'', [m, r]'''$, etc. denotetur idem, quod $[m, r]$ indicat, dummodo e serie $a, a', a'', a''',$ etc. resp. $a; a, a'; a, a', a'';$ etc. excludantur. e. g. $[5, 10]' = a_c^4 a^{vi} + a_c^3 (a'' a^v + a''' a''') + a_c^2 (a''^2 a'''' + a'' a''''^2) + a' a''^3 a''' + a_c^5$. $[3, 10]'' = a_c^2 a^{vi} + a'' (a''' a^v + a''''^2) + a''^2 a''''$. E quibus exemplis apparet, evolutionem quantitatum $[m, r], [m, r]'$ etc. tantummodo a numero quocunque integro r in suas partes discernendo pendere.

§. 2.

Theorema. Statuendo $a + a'x + a''x^2 + a'''x^3 + \text{etc.} = z$,
erit $z_c^n = [n, 0] + [n, 1]x + [n, 2]x^2 + [n, 3]x^3 + [n, 4]x^4 + \text{etc.}$

Demonstr. Sit $a' + a''x + a'''x^2 + a''''x^3 + \text{etc.} = z'$
erit $z_c^n = (a + xz')_c^n$ et (disq. II §. 5)
 $= a_c^n + a_c^{n-1}z'x + a_c^{n-2}z_c'^2x^2 + a_c^{n-3}z_c'^3x^3 + a_c^{n-4}z_c'^4x^4 \dots + z_c'^nx^n$
atque, statuendo

$$z' = z'_{\kappa 0} + z'_{\kappa 1}x + z'_{\kappa 2}x^2 + z'_{\kappa 3}x^3 + z'_{\kappa 4}x^4 + \text{etc.}$$

$$z_c'^2 = z_c'^2_{\kappa 0} + z_c'^2_{\kappa 1}x + z_c'^2_{\kappa 2}x^2 + z_c'^2_{\kappa 3}x^3 + z_c'^2_{\kappa 4}x^4 + \text{etc.}$$

$$z_c'^3 = z_c'^3_{\kappa 0} + z_c'^3_{\kappa 1}x + z_c'^3_{\kappa 2}x^2 + z_c'^3_{\kappa 3}x^3 + z_c'^3_{\kappa 4}x^4 + \text{etc.}$$

etc. etc. etc.

$$= a_c^n + a_c^{n-1}z'_{\kappa 0}x + a_c^{n-1}z'_{\kappa 1}x^2 + a_c^{n-1}z'_{\kappa 2}x^3 + a_c^{n-1}z'_{\kappa 3}x^4 + \text{etc.} \\ + a_c^{n-2}z_c'^2_{\kappa 0} + a_c^{n-2}z_c'^2_{\kappa 1}x + a_c^{n-2}z_c'^2_{\kappa 2}x^2 + a_c^{n-2}z_c'^2_{\kappa 3}x^3 + a_c^{n-2}z_c'^2_{\kappa 4}x^4 + \text{etc.} \\ + a_c^{n-3}z_c'^3_{\kappa 0} + a_c^{n-3}z_c'^3_{\kappa 1}x + a_c^{n-3}z_c'^3_{\kappa 2}x^2 + a_c^{n-3}z_c'^3_{\kappa 3}x^3 + a_c^{n-3}z_c'^3_{\kappa 4}x^4 + \text{etc.}$$

ita ut sit coefficientis generalis $z_c^n \kappa r$

$$= a_c^{n-1}z'_{\kappa}(r-1) + a_c^{n-2}z_c'^2_{\kappa}(r-2) + a_c^{n-3}z_c'^3_{\kappa}(r-3) \dots + z_c'^n_{\kappa}(r-n)$$

$$\text{aut} = a_c^{n-1}z'_{\kappa}(r-1) + a_c^{n-2}z_c'^2_{\kappa}(r-2) + a_c^{n-3}z_c'^3_{\kappa}(r-3) \dots + a_c^{n-r}z_c'^r_{\kappa 0}.$$

prout est $r >$ vel non $> n$. Nec non cuique facile persuaderi potest, significatione formarum $[m, r]$, $[m, r]'$ rite intellecta, oportere

$$\text{ut sit } [n, r] = a_c^{n-1}[1, r]' + a_c^{n-2}[2, r]' + a_c^{n-3}[3, r]' \dots + a[n-1, r] + [n, r]'$$

$$\text{aut} = a_c^{n-1}[1, r]' + a_c^{n-2}[2, r]' + a_c^{n-3}[3, r]' \dots + a_c^{n-r}[r, r]'$$

Jam ponamus pro omnibus valoribus ipsius $s < r$ esse $z_c^n \kappa s = [n, s]$

$$\text{adeoque etiam } z_c'^n \kappa s = [n, n + s]', \text{ erit } z'_{\kappa}(r-1) = [1, r]',$$

$$z_c'^2_{\kappa}(r-2) = [2, r]', z_c'^3_{\kappa}(r-3) = [3, r]', \text{ etc. igiturque } z_c^n \kappa r = a_c^{n-1}[1, r]'$$

$$+ a_c^{n-2}[2, r]' + a_c^{n-3}[3, r]' + \text{etc.} = [n, r].$$

$$\text{Hinc } z_c^n \kappa_0 = a_c^n = [n, 0]$$

$$z_c^n \kappa_1 = a_c^{n-1} z_c' \kappa_0 = [n, 1]$$

$$z_c^n \kappa_2 = a_c^{n-1} z_c' \kappa_1 + a_c^{n-2} z_c'^2 \kappa_0 = [n, 2]$$

$$z_c^n \kappa_3 = a_c^{n-1} z_c'^3 \kappa_2 + a_c^{n-2} z_c'^2 \kappa_1 + a_c^{n-3} z_c'^3 \kappa_0 = [n, 3]$$

$$\text{et generaliter } z_c^n \kappa_r = [n, r]$$

$$\text{Unde } z_c^n = z_c^n \kappa_0 + z_c^n \kappa_1 \cdot x + z_c^n \kappa_2 \cdot x^2 + z_c^n \kappa_3 \cdot x^3 + \text{etc.}$$

$$= [n, 0] + [n, 1]x + [n, 2]x^2 + [n, 3]x^3 + \text{etc.}$$

§. 3.

$$\text{Coroll. Quodsi } Z = A + A'y + A''y^2 + A'''y^3 + \text{etc.}$$

$$\text{et } y = a'x + a''x^2 + a'''x^3 + a''''x^4 + \text{etc.} = xz' \text{ erit etiam}$$

$$Z = A + A'xz' + A''y^2 z_c'^2 + A'''x^3 z_c'^3 + \text{etc. et, quia}$$

$$z_c'^n = [n, n]' + [n, n+1]'x + [n, n+2]'x^2 + [n, n+3]'x^3 + \text{etc.}$$

$$\text{habebitur } Z = A$$

$$\begin{aligned} &+ A'[1, 1]'x + A'[1, 2]'x^2 + A'[1, 3]'x^3 + A'[1, 4]'x^4 \\ &\quad + A''[2, 2]' \quad + A''[2, 3]' \quad + A''[2, 4]' \\ &\quad \quad + A'''[3, 3]' \quad + A'''[3, 4]' \\ &\quad \quad \quad + A''''[4, 4]' \end{aligned}$$

$$\text{E. g. statuendo } z = a + a'x + a''x^2 + a'''x^3 + \text{etc. erit pro quocunque valore ipsius } n \text{ (disq. II. §. 11)}$$

$$z^n = (a + y)^n = a^n + na^{n-1}y + n^2 a^{n-2}y^2 + n^3 a^{n-3}y^3 + \text{etc.}$$

$$= a^n + na^{n-1}y + n^2 a^{n-2}y^2 + n^3 a^{n-3}y^3 + \text{etc.}$$

$$\begin{aligned} &= a^n + na^{n-1}[1, 1]'x + n^2 a^{n-2}[1, 2]'x^2 + n^3 a^{n-3}[1, 3]'x^3 + n^4 a^{n-4}[1, 4]'x^4 + \text{etc.} \\ &\quad + n^2 a^{n-2}[2, 2]' \quad + n^3 a^{n-3}[2, 3]' \quad + n^4 a^{n-4}[2, 4]' \\ &\quad \quad + n^3 a^{n-3}[3, 3]' \quad + n^4 a^{n-4}[3, 4]' \\ &\quad \quad \quad + n^4 a^{n-4}[4, 4]' \end{aligned}$$

Simili modo, existente $y = a'x + a''x^2 + a'''x^3 + \text{etc.}$ $\log(1 + y)$. Siny , Cosy et aliae functiones ipsius y per series formae $A + A'x + A''x^2 + \text{etc.}$ exhiberi possunt.

§. 4.

Definitio. Quodsi series $B + B'x + B''x^2 + B'''x^3 + \text{etc.} = v$ a serie $A + A'x + A''x^2 + A'''x^3 + \text{etc.} = u$ ita pendeat, ut sit $B = A'$, $B' = 2A''$, $B'' = 3A'''$, $B''' = 4A''''$, etc. et generaliter $v \kappa n = (n + 1)u \kappa (n + 1)$ functio v dicatur *derivata* functionis u et per Du designetur.]

§. 5

Theorema. Statuendo $a + a'x + a''x^2 + a'''x^3 + \text{etc.} = y$ et $\alpha + \alpha'x + \alpha''x^2 + \alpha'''x^3 + \text{etc.} = z$ erit $Dyz = zDy + yDz$.

Demonstr. Cum sit $Dy = a' + 2a''x + 3a'''x^2 + 4a''''x^3 + \text{etc.}$ et $Dz = \alpha' + 2\alpha''x + 3\alpha'''x^2 + 4\alpha''''x^3 + \text{etc.}$ facile intelligitur fore $(zDy) \kappa n = a' \alpha^{(n)} + a^{(n+1)} 2a'' \alpha^{(n-1)} + 3a''' \alpha^{(n-2)} \dots + (n+1)a^{(n+1)} \alpha$
 $(yDz) \kappa n = (n+1)a \alpha^{(n+1)} + na' \alpha^{(n)} + (n-1)a'' \alpha^{(n-1)} \dots + a^{(n)} \alpha'$
 adeoque $(zDy + yDz) \kappa n$
 $= (n+1)(a \alpha^{(n+1)} + a' \alpha^{(n)} + a'' \alpha^{(n-1)} + a''' \alpha^{(n-2)} \dots + a^{(n+1)} \alpha) = (n+1)yz \kappa (n+1)$
 et $yDz + zDy = (yz) \kappa 1 + 2(yz) \kappa 2.x + 3(yz) \kappa 3.x^2 + 4(yz) \kappa 4.x^3 + \text{etc.}$
 $= D.yz.$

§. 6.

Coroll. Statuendo $z = y^n$, erit $D.yz = Dy^{n+1} = yDy^n + y^n Dy$ adeoque $Dy^2 = 2yDy$, $Dy^3 = yDy^2 + y^2Dy = 3y^2Dy$, $Dy^4 = yDy^3 + y^3Dy$

$= 4y^3Dy$ et generaliter $D.y^n = ny^{n-1}Dy$ pro quocunque valore positivo numeri n .

§. 7.

Theorema. Denotando per y ut antea seriem formae $a + a'x + a''x^2 + a'''x^3 + \text{etc.}$ erit, pro quocunque valore ipsius n , $D.y^n = ny^{n-1}Dy$.

Demonstr. Sit $a' + a''x + a'''x^2 + \text{etc.} = y'$, erit $y^{n-1} = (a + xy')^{n-1} = (\text{disq. II §. 11}) a^{n-1} + (n-1)a^{n-2}xy' + (n-1)_c^2 a^{n-3}x^2y'^2 + (n-1)_c^3 a^{n-4}x^3y'^3 + \text{etc.}$ et $ny^{n-1}Dy = na^{n-1}Dy + 2n_c^2 a^{n-2}xy'Dy + 3n_c^3 a^{n-3}x^2y'^2Dy + 4n_c^4 a^{n-4}x^3y'^3Dy + \text{etc.}$ hinc $n(y^{n-1}Dy)_{\kappa r} = na^{n-1}Dy_{\kappa r} + 2n_c^2 a^{n-2}(y'Dy)_{\kappa(r-1)} + 3n_c^3 a^{n-3}(y'^2Dy)_{\kappa(r-2)} + \dots + (r+1)n_c^{r+1} a^{n-r-1}(y'^rDy)_{\kappa 0}$. Est autem, μ numerum positivum integrum denotante, $y'^\mu Dy = y'^\mu(y' + xDy') = y'^{\mu+1} + xy'^\mu Dy' = y'^{\mu+1} + \frac{x}{\mu+1}Dy'^{\mu+1}$. Unde $(y'^\mu Dy)_{\kappa m} = y'^{\mu+1}_{\kappa m} + \frac{Dy'^{\mu+1}_{\kappa(m-1)}}{\mu+1} = y'^{\mu+1}_{\kappa m} + \frac{m}{\mu+1}y'^{\mu+1}_{\kappa m} = \frac{m+\mu+1}{\mu+1}y'^{\mu+1}_{\kappa m}$ adeoque $Dy_{\kappa r} = \frac{r+1}{1}y'_{\kappa r}$, $(y'Dy)_{\kappa(r-1)} = \frac{r+1}{2}y'^2_{\kappa(r-1)}$, $(y'^2Dy)_{\kappa(r-2)} = \frac{r+1}{3}y'^3_{\kappa(r-2)}$, etc. quibus valoribus substitutis, habebitur $n(y^{n-1}Dy)_{\kappa r} = (r+1)(na^{n-1}y_{\kappa r} + n_c^2 a^{n-2}y'^2_{\kappa(r-1)} + n_c^3 a^{n-3}y'^3_{\kappa(r-2)} + \dots + n_c^{r+1} a^{n-r-1}y'^{r+1}_{\kappa 0}) = (r+1)(a^n + na^{n-1}y'x + n_c^2 a^{n-2}y'^2x^2 + n_c^3 a^{n-3}y'^3x^3 + \text{etc.})_{\kappa(r+1)} = (r+1)(a + y'x)^n_{\kappa(r+1)} = (r+1)y^n_{\kappa(r+1)} = Dy^n_{\kappa r}$. Unde statim sequitur $ny^{n-1}Dy = Dy^n$ q. e. d.

§. 8.

Coroll. Statuendo $y^n = A + A'x + A''x^2 + A'''x^3 + A''''x^4 + \text{etc.}$

erit $(ny^n Dy) \kappa r = n a' A^{(r)} + 2n a'' A^{(r-1)} + 3n a''' A^{(r-2)} + 4n a'''' A^{(r-3)} + \dots + (r+1) n a^{(r+1)} A$; $(y D. y^n) \kappa r = (r+1) a A^{(r+1)} + r a' A^{(r)} + (r-1) a'' A^{(r-1)} + \dots + a^r A'$ et quia $(ny^n Dy) \kappa r = (y D. y^n) \kappa r$ habebitur $(r+1) a A^{(r+1)} = (n-r) a' A^{(r)} + (2n-r+1) a'' A^{(r-1)} + (3n-r+2) a''' A^{(r-2)} + \dots + (r+1) n a^{(r+1)} A$, cujus aequationis ope omnes coefficientes cujuscunque potestatis polynomii aut infinitomii y^n successive determinari possunt.

§. 9.

Lemma. Denotantibus r, n tales numeros positivos, ut sit $r > n$ et existente $z = a + a'x + a''x^2 + a'''x^3 + \text{etc.}$ erit $(z^{-r} D. z^n x^n) \kappa (r-1) = 0$.

Demonstr. Quodsi statuimus $z^n = A + A'x + A''x^2 + A'''x^3 + \text{etc.}$ erit $z^n x^n = Ax^n + A'x^{n+1} + A''x^{n+2} + A'''x^{n+3} + \text{etc.}$ et

$$D. z^n x^n = nAx^{n-1} + (n+1)A'x^n + (n+2)A''x^{n+1} + (n+3)A'''x^{n+2} + \text{etc.} \\ = nx^{n-1}z^n + x^n D z^n = nx^{n-1}z^{n-1} (z + x D z) \text{ adeoque}$$

$$z^{-r} D. x^n z^n = nx^{n-1} (z^{n-r} + x z^{n-r-1} D z) = nx^{n-1} \left(z^{n-r} + \frac{x D z^{n-r}}{n-r} \right) \text{ et}$$

$$(z^{-r} D. z^n x^n) \kappa (r-1) = n \left(z^{n-r} \kappa (r-n) + \frac{D z^{n-r}}{n-r} \kappa (r-n-1) \right) = n (z^{n-r} \kappa (r-n) - z^{n-r} \kappa (r-n)) = 0.$$

§. 10.

Problema. Existente $y = ax + a'x^2 + a''x^3 + a'''x^4 + \text{etc.} = xy'$ functionem datam $z = A + A'x + A''x^2 + A'''x^3 + \text{etc.}$ in seriem $\mathcal{A} + \mathcal{A}'y + \mathcal{A}''y^2 + \mathcal{A}'''y^3 + \mathcal{A}''''y^4 + \text{etc.}$ transmutare.

Solutio. Quia $A + A'x + A''x^2 + A'''x^3 + \text{etc.} = \mathcal{A} + \mathcal{A}'y x + \mathcal{A}''y'^2 x^2 + \mathcal{A}'''y'^3 x^3 + \text{etc.}$ oportet esse

$$A' = \mathcal{A}'(xy') \kappa 1$$

$$A'' = \mathcal{A}'(xy') \kappa 2 + \mathcal{A}''(x^2y'^2) \kappa 2$$

$$A''' = \mathcal{A}'(xy') \kappa 3 + \mathcal{A}''(x^2y'^2) \kappa 3 + \mathcal{A}'''(x^3y'^3) \kappa 3$$

$$A'''' = \mathcal{A}'(xy') \kappa 4 + \mathcal{A}''(x^2y'^2) \kappa 4 + \mathcal{A}'''(x^3y'^3) \kappa 4 + \mathcal{A}''''(x^4y'^4) \kappa 4$$

$$\vdots$$

$$A^{(n)} = \mathcal{A}'(xy') \kappa n + \mathcal{A}''(x^2y'^2) \kappa n + \mathcal{A}'''(x^3y'^3) \kappa n \dots + \mathcal{A}^{(n)}(x^ny'^n) \kappa n$$

Quarum aequationem primam, secundam, tertiam etc. respective per $y'^{-n} \kappa (n-1)$, $2 y'^{-n} \kappa (n-2)$, $3 x'^{-n} \kappa (n-3)$, ... $n y'^{n-1} \kappa 0$ multiplicando productaque summando nanciscemur aequationem

$$(y'^{-n} Dz) \kappa (n-1) = \mathcal{A}'(y'^{-n} Dxy') \kappa (n-1) + \mathcal{A}''(y'^{-n} Dx^2y'^2) \kappa (n-1) + \mathcal{A}'''(y'^{-n} D. x^3y'^3) \kappa (n-1) \dots + \mathcal{A}^{(n)}(y'^{-n} D. x^ny'^n) \kappa (n-1)$$

$$= (\S. 9) \mathcal{A}^{(n)} y'^{-n} D. x^ny'^n \kappa n-1$$

$$= \mathcal{A}^{(n)} [y'^{-n} (ny'^n \kappa 0. x^{n-1} + (n+1)y'^n \kappa 1. x^n + (n+2)y'^n \kappa 2. x^{n+1} + \text{etc.})] \kappa (n-1)$$

$$= n \mathcal{A}^{(n)}. y'^{-n} \kappa 0. y'^n \kappa 0 = n \mathcal{A}^{(n)}, \text{ ergo } \mathcal{A}^{(n)} = \frac{1}{n} (y'^{-n} Dz) \kappa (n-1) \text{ et}$$

$$z = A + (y'^{-1} Dz) \kappa 0 y + \frac{1}{2} y'^{-2} Dz \kappa 1. y^2 + \frac{1}{3} (y'^{-3} Dz) \kappa 2. y^3 + \frac{1}{4} (y'^{-4} Dz) \kappa 3 y^4 + \&c.$$

§. 11.

Coroll. Cum sit $y = xy'$ adeoque $\left(\frac{x}{y}\right)^m = y'^{-m}$, habebitur, y'^{-m} loco ipsius z substituto, pro quocunque valore ipsius m

$$\left(\frac{x}{y}\right)^m = y'^{-m} \kappa 0 + (y'^{-1} Dy'^{-m}) \kappa 0. y + \frac{1}{2} (y'^{-2} Dy'^{-m}) \kappa 1. y^2 + \frac{1}{3} (y'^{-3} Dy'^{-m}) \kappa 2. y^3 + \text{etc.}$$

et quia $y'^{-r} Dy'^{-m} = -m y'^{-(r+m)} Dy' = \frac{m}{r+m} D. y'^{-(r+m)}$ adeoque

$$(y'^{-r} Dy'^{-m}) \kappa (r-1) = \frac{m}{r+m} Dy'^{-(r+m)} \kappa (r-1) = \frac{mr}{r+m} y'^{-(r+m)} \kappa r, \text{ erit}$$

$$\left(\frac{x}{y}\right)^m = y'^{-m} \kappa 0 + \frac{m}{m+1} y'^{-(m+1)} \kappa 1. y + \frac{m}{m+2} y'^{-(m+2)} \kappa 2. y^2 + \frac{m}{m+3} y'^{-(m+3)} \kappa 3. y^3$$

$$+ \text{etc. et } x^m = \frac{m}{m} y'^{-m} \kappa_0 \cdot y^m + \frac{m}{m+1} y'^{-(m+1)} \kappa_1 \cdot y^{m+1} + \frac{m}{m+3} y'^{-(m+2)} \kappa_2 \cdot y^{m+2} \\ + \frac{m}{m+3} y'^{-(m+3)} \kappa_3 \cdot y^{m+3} + \text{etc.}$$

§. 12.

Coroll. Sit $y = x(1+x)^{-\theta}$ et, pro quocunque valore ipsius m , $z = (1+x)^m$, erit $Dz = m(1+x)^{m-1}$, $y'^{-1} = (1+x)^\theta$ et $\frac{1}{r}(y'^{-r}Dz) \kappa(r-1) = \frac{m}{r} (1+x)^{m+r\theta-1} \kappa(r-1) = \frac{m}{r} (m+r\theta-1)_c^{r-1}$ adeoque

$$(1+x)^m = 1 + my + \frac{m}{2} (m+2\theta-1)_c y^2 + \frac{m}{3} (m+3\theta-1)_c^2 y^3 + \frac{m}{4} (m+4\theta-1)_c^3 y^4$$

+ etc. Substituendo w , z , x resp. loco ipsarum y , $1+x$, m , aequatio $y = x(1+x)^{-\theta}$ transmutatur in $wz^\theta - z + 1 = 0$

$$\text{et erit } z_c^n = 1 + xw + \frac{x}{2} (x+2\theta-1)_c w + \frac{x}{3} (x+3\theta-1)_c^2 w^2 + \text{etc.}$$

quam aequationem alio modo (disq. II. §. 17) invenimus.

§. 13.

Theorema. Statuendo $(1+\alpha x)(1+\alpha'x)(1+\alpha''x)(1+\alpha'''x)(1+\alpha''''x)$ etc. $= 1 + ax + a'x^2 + a''x^3 + a'''x^4 + \text{etc.} = z$, $\alpha + \alpha' + \alpha'' + \alpha''' + \text{etc.} = s$; $\frac{1}{2}(\alpha^2 + \alpha'^2 + \alpha''^2 + \alpha'''^2 + \text{etc.}) = s'$; $\frac{1}{3}(\alpha^3 + \alpha'^3 + \alpha''^3 + \alpha'''^3 + \text{etc.}) = s''$; et generaliter $\frac{1}{n}(\alpha^n + \alpha'^n + \alpha''^n + \alpha'''^n + \text{etc.}) = s^{(n-1)}$ atque

per (n, r) idem respectu quantitatum $s, s', s'', s''', \text{etc.}$ denotando, quod (§. 1) $[n, r]$ respectu seriei $a, a', a'', a''' \text{etc.}$ indicat, erit

$$s^{(n)} = n^{(n)}[n+1, 0] - (n-1)^{(n-1)}[n, 1] + (n-2)^{(n-2)}[n-1, 2] \dots \pm [1, n]$$

$$\text{et } a^{(n)} = (n+1, 0) - (n, 1) + (n-1, 2) - (n-2, 3) \dots \pm (1, n).$$

Demonstr. I. Ponamus $a + a'x + a''x^2 + a'''x^3 + \text{etc.} = y$
erit $z = 1 + yx$ et (disq. II. §. 12)

$$\log z = yx - \frac{1}{2} y^2 x^2 + \frac{1}{3} y^3 x^3 - \frac{1}{4} y^4 x^4 + \frac{1}{5} y^5 x^5 - \text{etc.}$$

$$= yx - y_c^2 x^2 + 2^{1/2} y_c^3 x^3 - 3^{1/3} y_c^4 x^4 + 4^{1/4} y_c^5 x^5 - \text{etc.}$$

Nec non erit, quia $z = (1 + \alpha x)(1 + \alpha' x)(1 + \alpha'' x)(1 + \alpha''' x) \text{ etc.}$

$$\log z = sx - s'x^2 + s''x^3 - s'''x^4 + s''''x^5 - \text{etc.}$$

$$\text{itaque } s^{(n)} = n^{(n)} y^{n+1} \kappa_0 - (n-1)^{(n-1)} y_c^n \kappa_1 + (n-2)^{(n-2)} y_c^{n-1} \kappa_2 \dots + y \kappa_n$$

$$= (\S. 2) n^{(n)} [n+1, 0] - (n-1)^{(n-1)} [n, 1] + [n-2]^{(n-2)} [n, 2] \dots + [1, n].$$

II. Cum sit $sx - s'x^2 + s''x^3 - s'''x^4 + \text{etc.} = \log(1 + ax + a'x^2 + a''x^3 + \text{etc.})$
adeoque $(sx + s'x^2 + s''x^3 + s'''x^4 + \text{etc.}) = \log(1 - ax + a'x^2 - a''x^3 + \text{etc.})$
erit, statuendo $s + s'x + s''x^2 + s'''x^3 + \text{etc.} = u,$

$$1 - ax + a'x^2 - a''x^3 + \text{etc.} = e^{-ux} = 1 - ux + u^2 x^2 - u^3 x^3 + \text{etc.}$$

$$\text{Unde } a^{(n)} = u_c^{n+1} \kappa_0 - u_c^n \kappa_1 + u_c^{n-1} \kappa_2 \dots + u \kappa_n$$

$$= (\S. 2.) (n+1, 0) - (n, 1) + (n-1, 2) - (n-2, 3) \dots + (1, n).$$

§. 14.

Theorema. Denotatione signorum (§. prec) salva habebitur, statuendo $s S, 2s' = S', 3s'' = S'', 4s''' = S''', \text{etc.}$

$$S^{(r)} - a S^{(r-1)} + a'' S^{(r-3)} \dots + a^{(r-1)} S + (r+1) a^{(r)} = 0$$

$$\text{Demonstr. Sit } \frac{a + 2a'x + 3a''x^2 + 4a'''x^3 + 5a''''x^4 + \text{etc.}}{1 + ax + a'x^2 + a''x^3 + \text{etc.}} = z$$

$$\text{et } a + a'x + a''x^2 + a'''x^3 + \text{etc.} = y \text{ erit } z = \frac{y + x Dy}{1 + xy}$$

$$= y - xy^2 + x^2 y^3 - x^3 y^4 + x^4 y^5 - x^5 y^6 + \text{etc.}$$

$$+ xDy - x^2 yDy + x^3 y^2 Dy - x^4 y^3 Dy + x^5 y^4 Dy - \text{etc.}$$

$$= y + xDy - x(y^2 + \frac{1}{2} xD.y^2) + x^2(y^3 + \frac{1}{3} xD.y^3) - x^3(y^4 + \frac{1}{4} xD.y^4) + \text{etc.}$$

$$\text{Est autem } x^{m-1}(y^m + \frac{1}{m} xD.y^m) \kappa r = y^m \kappa(r-m+1) + \frac{1}{m} D.y^m \kappa(r-m)$$

$$= y^m \kappa(r-m+1) + \frac{r-m+1}{m} y^m \kappa(r-m+1) = \frac{r+1}{m} y^m \kappa(r-m+1)$$

adeoque $z^r = (r+1)(y^r - \frac{1}{2}y^2r(r-1) + \frac{1}{3}y^3r(r-2) \dots + \frac{1}{r+1}y^{r+1}r0)$
 $= (\S. 13. I) \pm (r+1)s^{(r)} = \pm S^{(r)}$ et $z = S - S'x + S''x^2 - S'''x^3 + \text{etc.}$
 Oportet ergo ut sit $a + 2a'x + 3a''x^2 + 4a'''x^3 + a''''x^4 + \text{etc.}$
 $= (1 + ax + a'x^2 + a''x^3 + \text{etc.}) (S - S'x + S''x^2 - S'''x^3 + \text{etc.})$
 Itaque erit $a = S$; $2a' = aS - S'$; $3a'' = a'S - aS' + S''$; etc.
 adeoque $S^{(r)} = aS^{(r-1)} - a'S^{(r-2)} + a''S^{(r-3)} \dots \pm a^{(r-1)}S \mp (r+1)a^{(r)} = 0.$

§. 15.

Loco quantitatum $a, a', a'', a''', \text{etc.}$ resp. $\frac{1}{1.2.3}, \frac{1}{1.2.3.4.6}, \frac{1}{1.2.3.4.5.6.7}, \text{etc.}$
 substitutis erit $S = \frac{1}{6}, S' = \frac{1}{6} \cdot \frac{1}{6} - \frac{1^2}{1.2.6} = \frac{1}{90}$
 $S'' = \frac{1}{6} \cdot \frac{1}{90} - \frac{1}{1.2.6} \cdot \frac{1}{6} + \frac{1^3}{3.4.6} = \frac{1}{945}$
 $S''' = \frac{1}{6} \cdot \frac{1}{945} - \frac{1}{1.2.6} \cdot \frac{1}{90} + \frac{1}{3.4.6} \cdot \frac{1}{6} - \frac{1^4}{3.4.6.7.2} = \frac{1}{9450}$
 $S'''' = \frac{1}{6} \cdot \frac{1}{9450} - \frac{1}{1.2.6} \cdot \frac{1}{945} + \frac{1}{3.4.6} \cdot \frac{1}{90} - \frac{1}{3.4.6.7.2} \cdot \frac{1}{6} + \frac{1^5}{3.4.6.7.2.1.10} = \frac{1}{93555} \text{ etc.}$
 adeoque, quia (disq. 1. §. 16) $(1+x)(1+\frac{1}{4}x)(1+\frac{1}{9}x)(1+\frac{1}{16}x) \text{ etc.}$
 $= 1 + \frac{1}{3}\pi^2x + \frac{1}{5}\pi^4x^2 + \frac{1}{7}\pi^6x^3 + \text{etc.},$ erit $1 + \frac{1}{4} + \frac{1}{9} + \text{etc.} = \frac{\pi^2}{6};$
 $1 + \frac{1}{4^2} + \frac{1}{9^2} + \text{etc.} = \frac{\pi^4}{90}; 1 + \frac{1}{4^3} + \frac{1}{9^3} + \text{etc.} = \frac{\pi^6}{945}; 1 + \frac{1}{4^4} + \frac{1}{9^4} + \text{etc.} = \frac{\pi^8}{9450};$
 $1 + \frac{1}{4^5} + \frac{1}{9^5} + \text{etc.} = \frac{\pi^{10}}{93555}; \text{ etc.}$ Simili modo colligimus ex aequatione
 $(1+x)(1+\frac{1}{9}x)(1+\frac{1}{25}x)(1+\frac{1}{49}x) \text{ etc.} = 1 + (\frac{\pi}{2})^2x + (\frac{\pi}{2})^4x^2 + (\frac{\pi}{2})^6x^3 + \text{etc.}$
 $1 + \frac{1}{9} + \frac{1}{25} + \text{etc.} = \frac{1}{2}(\frac{\pi}{2})^2; 1 + \frac{1}{9^2} + \frac{1}{25^2} + \text{etc.} = \frac{1}{8}(\frac{\pi}{2})^4; 1 + \frac{1}{9^3} + \frac{1}{25^3} + \text{etc.}$
 $= \frac{1}{15}(\frac{\pi}{2})^6; 1 + \frac{1}{9^4} + \frac{1}{25^4} + \text{etc.} = \frac{1}{630}(\frac{\pi}{2})^8; 1 + \frac{1}{9^5} + \frac{1}{25^5} + \text{etc.} = \frac{1}{2835}(\frac{\pi}{2})^{10} \text{ etc.}$
 (Vide Euleri Intr. in anal. inf. (§. 168).)

DISQUISITIO QUARTA

DE AEQUATIONIBUS CURVARUM ET SUPERFICIERUM SECUNDI GRADUS TRANSFORMANDIS.

§. I.

Theorema. Si, $x, y; x', y'$ coordinatas cujuscunque puncti respectu duorum systematum axium inter se orthogonalium denotantibus, aequatio

$$(L) \quad Ax^2 + By^2 + 2Cxy + 2Ex + 2Fy + G = 0$$

per substitutionem (1) $x = \xi x' + \zeta y' + p, y = \eta x' + \eta' y' + q$ in aequationem

$$(M) \quad \mathcal{A}x'^2 + \mathcal{B}y'^2 + 2\mathcal{C}x'y' + 2\mathcal{E}x' + 2\mathcal{F}y' + \mathcal{G} = 0$$

transformetur, inter coefficientes aequationum (L) (M) sequentes tres aequationes 1) $\mathcal{A} + \mathcal{B} = A + B,$

2) $\mathcal{A}\mathcal{B} = \mathcal{C}^2 = AB - C^2;$ et 3) $\mathcal{B}\mathcal{E}^2 + \mathcal{A}\mathcal{F}^2 - 2\mathcal{C}\mathcal{E}\mathcal{F} - (\mathcal{A}\mathcal{B} - \mathcal{C}^2)\mathcal{G} = BE^2 + AF^2 - 2CEF - (AB - C^2)G$ locum habeant necesse est.

Demonstr. I. Nullo negotio apparet oportere esse $(x - p)^2 + (y - q)^2 = x'^2 + y'^2$ adeoque propter (1)

$$(\xi^2 + \eta^2)x'^2 + (\zeta^2 + \eta'^2)y'^2 + 2(\xi\zeta' + \eta\eta')x'y' = x'^2 + y'^2$$

unde statim hae tres aequationes (2) $\xi^2 + \eta^2 = 1, \zeta^2 + \eta'^2 = 1,$

$\xi\zeta' + \eta\eta' = 0$ sequuntur, quarum ope ex aequationibus

$$\xi x' + \zeta y' = x - p; \eta x' + \eta' y' = y - q$$

evadunt $x' = \xi(x-p) + \eta(y-q)$, $y' = \xi'(x-p) + \eta'(y-q)$ Hinc
 $x'^2 + y'^2 = (\xi^2 + \xi'^2)(x-p)^2 + (\eta^2 + \eta'^2)(y-q)^2 + 2(\xi\eta + \xi'\eta')(x-p)(y-q)$
 $= (x-p)^2 + (y-q)^2$ atque (3) $\xi^2 + \xi'^2 = 1$, $\eta^2 + \eta'^2 = 1$, $\xi\eta + \xi'\eta' = 0$

II. Quia aequatio (L) per substitutionem (1) in (M) transmutatur, oportet ut sit

$$(4) \begin{cases} \mathcal{A} = A\xi^2 + B\eta^2 + 2C\xi\eta \\ \mathcal{B} = A\xi'^2 + B\eta'^2 + 2C\xi'\eta' \\ \mathcal{C} = A\xi\xi' + B\eta\eta' + C(\xi\eta' + \xi'\eta) \end{cases}$$

$$(5) \begin{cases} \mathcal{E} = A\xi p + B\eta q + C(\xi q + \eta p) + E\xi + F\eta \\ \mathcal{F} = A\xi' p + B\eta' q + C(\xi' q + \eta' p) + E\xi' + F\eta' \end{cases}$$

$$(6) \quad \mathcal{G} = A p^2 + B q^2 + 2C p q + 2E p + 2F q + G.$$

Ex duabus primis aequationum (4) propter aeqq. (3) statim sequitur $\mathcal{A} + \mathcal{B} = A + B$. Erit ergo etiam, aeqq. (2) adhibitis,
 $\mathcal{B} = A + B - \mathcal{A} = A + B - (A\xi^2 + B\eta^2 + 2C\xi\eta) = A\eta^2 + B\xi^2 - 2C\xi\eta$
 $\mathcal{A} = A + B - \mathcal{B} = A + B - (A\xi'^2 + B\eta'^2 + 2C\xi'\eta') = A\eta'^2 + B\xi'^2 - 2C\xi'\eta'$
 $-\mathcal{C} = -A\xi\xi' - B\eta\eta' - C(\xi\eta' + \xi'\eta) = A\eta\eta' + B\xi\xi' - C(\xi\eta' + \xi'\eta)$

III. Statuendo (8) $A\xi + C\eta = P$, $B\eta + C\xi = Q$,
 $A\xi' + C\eta' = P'$, $B\eta' + C\xi' = Q'$ erit

$$(7) \begin{cases} \mathcal{A} = P\xi + Q\eta, \mathcal{B} = P'\xi' + Q'\eta' \\ \mathcal{C} = P\xi' + Q\eta' = P'\xi + Q'\eta \end{cases}$$

$$(8) \begin{cases} \mathcal{E} = Pp + Qq + E\xi + F\eta \\ \mathcal{F} = P'p + Q'q + E\xi' + F\eta' \end{cases}$$

Nec non ex aeqq. (7) facile deducitur

$$(9) \begin{cases} \mathcal{A}\xi + \mathcal{C}\xi' = P & \mathcal{B}\xi' + \mathcal{C}\xi = P' \\ \mathcal{A}\eta + \mathcal{C}\eta' = Q & \mathcal{B}\eta' + \mathcal{C}\eta = Q' \end{cases}$$

Itaque erit $PQ' - QP' = (AB - C^2)(\xi\eta' - \eta\xi') = (\mathcal{A}\mathcal{B} - \mathcal{C}^2)(\xi\eta' - \eta\xi')$
ergo $AB - C^2 = \mathcal{A}\mathcal{B} - \mathcal{C}^2$.

IV. Ex aeqq. (5) et (3) nanciscemur

$$(10) \begin{cases} \mathfrak{E}\xi + \mathfrak{F}\xi' = Ap + Cq + E \\ \mathfrak{E}\eta + \mathfrak{F}\eta' = Cp + Bq + F \end{cases}$$

adeoque

$$(11) \begin{cases} (AB - C^2)p = (\mathfrak{E}\xi + \mathfrak{F}\xi' - E)B - (\mathfrak{E}\eta + \mathfrak{F}\eta' - F)C \\ (AB - C^2)q = (\mathfrak{E}\eta + \mathfrak{F}\eta' - F)A - (\mathfrak{E}\xi + \mathfrak{F}\xi' - E)C \end{cases}$$

$$\begin{aligned} \text{Est autem (6) } \mathfrak{G} - G &= (Ap + Cq + 2E)p + (Bq + Cp + 2F)q \\ &= (10) (\mathfrak{E}\xi + \mathfrak{F}\xi' + E)p + (\mathfrak{E}\eta + \mathfrak{F}\eta' + F)q \end{aligned}$$

inde, denotando $AB - C^2 = \mathfrak{A}\mathfrak{B} - \mathfrak{C}^2$ per D,

$$\begin{aligned} D(\mathfrak{G} - G) &= ((\mathfrak{E}\xi + \mathfrak{F}\xi')^2 - E^2)B + ((\mathfrak{E}\eta + \mathfrak{F}\eta')^2 - F^2)A - 2((\mathfrak{E}\xi + \mathfrak{F}\xi') \\ &(\mathfrak{E}\eta + \mathfrak{F}\eta') - EF)C = \mathfrak{E}^2(B\xi^2 + A\eta^2 - 2C\xi\eta) + \mathfrak{F}^2(B\xi'^2 + A\eta'^2 - 2C\xi'\eta') \end{aligned}$$

$$+ 2\mathfrak{E}\mathfrak{F}(B\xi\xi' + A\eta\eta' - C(\xi\eta' + \eta\xi')) - (BE^2 + AF^2 - 2CEF)$$

$$= (6) \mathfrak{B}\mathfrak{E}^2 + \mathfrak{A}\mathfrak{F}^2 - 2\mathfrak{C}\mathfrak{E}\mathfrak{F} - (BE^2 + AF^2 - 2CEF)$$

$$\text{aut } \mathfrak{B}\mathfrak{E}^2 + \mathfrak{A}\mathfrak{F}^2 - 2\mathfrak{C}\mathfrak{E}\mathfrak{F} - (\mathfrak{A}\mathfrak{B} - \mathfrak{C}^2)\mathfrak{G} = BE^2 + AF^2 - 2CEF - (AB - C^2)G.$$

§. 2.

Problema. Aequationem generalem curvae secundi gradus (L) $Ax^2 + By^2 + 2Cxy + 2Ex + 2Fy + F = 0$ in simpliciores $\mathfrak{A}x^2 + \mathfrak{B}y^2 + 2\mathfrak{C}x = 0$ transmutare.

Solutio. Propter aequationes conditionales inter coefficientes oportet esse: 1) $\mathfrak{A} + \mathfrak{B} = A + B$ 2) $\mathfrak{A}\mathfrak{B} = AB - C^2$ et 3) $\mathfrak{B}\mathfrak{E}^2 = AF^2 + BE^2 - 2CEF - (AB - C^2)G$. E duabus primis aequationibus colligimus $\mathfrak{A}, \mathfrak{B}$ esse duas radices $\frac{1}{2}(A+B \pm \sqrt{(A-B)^2 + 4C^2})$ quas semper reales esse manifestum est. Coefficientis $\mathfrak{E}$ semper realis erit, si $AB - C^2$ est quantitas negativa, quia tunc $\mathfrak{A}, \mathfrak{B}$ heterogeneas esse patet, adeoque $\mathfrak{B}$ ita assumi potest, ut sit homo-

genea cum $AF^2 + BE^2 - 2CEF - (AB - C^2)G = K$. Sin autem $AB - C^2$ est positiva, $\mathcal{A}$ et $\mathcal{B}$ simul sunt positivae aut negativae. Quando ergo K cum $\mathcal{B}$ adeoque etiam cum $\mathcal{A} + \mathcal{B}$ aut $A + B$ non sit homogenea, $\mathcal{E}$ quantitas realis esse nequit.

Propter aequationes (7), (9), habebitur, quia $\mathcal{E} = 0$,

$$(A - \mathcal{A})\xi + C\eta = 0, \quad (A - \mathcal{B})\xi' + C\eta' = 0,$$

$$C\xi + (B - \mathcal{A})\eta = 0, \quad C\xi' + (B - \mathcal{B})\eta' = 0,$$

$$\text{inde } \xi^2 : \eta^2 = \mathcal{A} - B : \mathcal{A} - A \text{ et } \xi'^2 : \eta'^2 = (\mathcal{B} - B) : (\mathcal{B} - A)$$

$$\text{adeoque } \xi = \sqrt{\frac{\mathcal{A} - B}{\mathcal{A} - \mathcal{B}}}, \eta = \sqrt{\frac{\mathcal{A} - A}{\mathcal{A} - \mathcal{B}}}, \xi' = \sqrt{\frac{\mathcal{B} - B}{\mathcal{A} - \mathcal{B}}}, \eta' = \sqrt{\frac{\mathcal{B} - A}{\mathcal{B} - \mathcal{A}}}$$

Aequationum (11) ope, statuendo $\mathcal{F} = 0$, inveniuntur

$$p = \frac{(B\xi - C\eta)\mathcal{E} - EB + FC}{AB - C^2}, \quad q = \frac{(A\eta - C\xi)\mathcal{E} - FA + EC}{AB - C^2}$$

Ceterum facile perspicitur $\xi, \xi'; \eta, \eta'$ esse cosinus angulorum quos axes coordinatarum x', y' respective cum axibus x, y formant et p, q coordinatas puncti intersectionis axium x', y' respectu axium x, y . Statuendo $\xi = \cos \alpha$ erit $\eta = \sin \alpha$,

$$\xi' = \pm \sin \alpha, \eta' = \mp \cos \alpha \text{ adeoque } \tan 2\alpha = \frac{2\xi\eta}{\xi^2 - \eta^2} \\ = \frac{2\sqrt{(\mathcal{A}^2 - (A+B)\mathcal{A} + AB)}}{A - B} = \frac{2\sqrt{(AB - \mathcal{A}(A+B-\mathcal{A}))}}{A - B} = \frac{2\sqrt{(AB - \mathcal{A}\mathcal{B})}}{A - B} = \frac{2C}{A - B}.$$

§. 3.

Coroll. Curva cujus aequatio generalis (L) $Ax^2 + By^2 + 2Cxy + 2Ex + 2Fy + G = 0$, in formam simpliciores $\mathcal{A}x'^2 + \mathcal{B}y'^2 + 2\mathcal{E}'x = 0$ redigi potest, modo nec $\mathcal{B}$, nec $\mathcal{E}$ sit $= 0$, est sectio conica, et quidem ellipsis, hyperbola aut parabola, prout est $AB - C^2$ quantitas positiva, negativa aut $= 0$. Quodsi

est hyperbola, erit $Ax^2 + By^2 + 2Cxy + 2Ex + 2Fy + \frac{BE^2 + AF^2 - 2CEF}{AB - C^2}$

$= 0$, aequatio ejus asymptotarum. Aequationem generalem (L), modo ne sit $AB - C^2 = 0$, etiam in $\mathcal{A}x'^2 + \mathcal{B}y'^2 + \mathcal{G}' = 0$ transformari licere facile apparet. Tunc quantitates angulares ξ, η, ξ', η' valores modo inventos servant, reliquae autem ita determinantur

ut sit $p = \frac{FC - EB}{AB - C^2}$, $q = \frac{EC - FA}{AB - C^2}$, $\mathcal{G} = \frac{BE^2 + AF^2 - 2CEF}{AB - C^2} - G$.

§. 4.

Coroll. Quodsi functio $Ax^2 + By^2 + 2Cxy + 2Ex + 2Fy$ per substitutionem $x = \xi x' + \xi' y'$; $y = \eta x' + \eta' y'$ in similem $\mathcal{A}x'^2 + \mathcal{B}y'^2 + 2\mathcal{C}x'y' + 2\mathcal{E}x' + 2\mathcal{F}y'$ transeat, ad tres aequationes conditionales (§. 1) $\mathcal{A} + \mathcal{B} = A + B$; $\mathcal{A}\mathcal{B} - \mathcal{C}^2 = AB - C^2$; $\mathcal{A}\mathcal{F}^2 + \mathcal{B}\mathcal{E}^2 - 2\mathcal{C}\mathcal{E}\mathcal{F} = AF^2 + BE^2 - 2CEF$ adhuc quarta, ex aeqq. (§ 5) $\mathcal{E} = E\xi + F\eta$, $\mathcal{F} = E\xi' + F\eta'$ prodiens, $\mathcal{E}^2 + \mathcal{F}^2 = E^2 + F^2$ adjicienda est. Quantitatum ergo $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}, \mathcal{F}$ una sola ad libitum determinari licet. Ponamus $\mathcal{F} = 0$, erit, quia $\mathcal{E}^2 = E^2 + F^2$ et $\mathcal{B}\mathcal{E}^2 = AF^2 + BE^2 - 2CEF$, $\mathcal{B} = \frac{AF^2 + BE^2 - 2CEF}{E^2 + F^2}$, $\mathcal{A} = A + B - \mathcal{B} = \frac{AE^2 + BF^2 + 2CEF}{E^2 + F^2}$

$$\mathcal{C} = \sqrt{\mathcal{A}\mathcal{B} - AB + C^2} = \frac{(A - B)EF - C(E^2 - F^2)}{E^2 + F^2}$$

et aequationibus $\mathcal{E} = E\xi + F\eta$, $0 = E\xi' + F\eta'$ rite cum aeqq. (2) (3) (§. 1) combinatis, habebitur

$$\xi = \frac{E}{\sqrt{E^2 + F^2}}, \eta = \frac{F}{\sqrt{E^2 + F^2}}, \xi' = \frac{F}{\sqrt{E^2 + F^2}}, \eta' = -\frac{E}{\sqrt{E^2 + F^2}}$$

§. 5.

Theorema. Si aequatio $Ax^2 + By^2 + Cz^2 + 2Eyz + 2Fzx + 2Gxy + 2Hx + 2Jy + 2Kz + L = 0$ per substitutionem

$$(1) \begin{cases} x = \xi x' + \xi' y' + \xi'' z' + p \\ y = \eta x' + \eta' y' + \eta'' z' + q \\ z = \zeta x' + \zeta' y' + \zeta'' z' + r. \end{cases}$$

$x, y, z; x', y', z'$ coordinatas cujuscunque puncti inter se orthogonales denotantibus, in aliam similem (Q) $\mathcal{A}x'^2 + \mathcal{B}y'^2 + \mathcal{C}z'^2 + 2\mathcal{E}y'z' + 2\mathcal{F}z'x' + 2\mathcal{G}x'y' + 2\mathcal{H}x' + 2\mathcal{J}y' + 2\mathcal{K}z' + \mathcal{L} = 0$ transmutetur, oportet ut inter coefficients harum duarum aequationum, denotando $BC - E^2, CA - F^2, AB - G^2, FG - AE, GE - BF, EF - CG$ respective per A', B', C', E', F', G' et $\mathcal{B}\mathcal{C} - \mathcal{E}^2, \mathcal{C}\mathcal{A} - \mathcal{F}^2, \mathcal{A}\mathcal{B} - \mathcal{G}^2, \mathcal{F}\mathcal{G} - \mathcal{A}\mathcal{E}, \mathcal{G}\mathcal{E} - \mathcal{B}\mathcal{F}, \mathcal{E}\mathcal{F} - \mathcal{C}\mathcal{G}$ resp. per $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{E}', \mathcal{F}', \mathcal{G}'$, sequentes quatuor aequationes conditionales locum habeant:

$$\begin{aligned} 1) & \mathcal{A} + \mathcal{B} + \mathcal{C} = A + B + C; \quad 2) \mathcal{A}' + \mathcal{B}' + \mathcal{C}' = A' + B' + C' \\ 3) & \mathcal{A}\mathcal{E}^2 + \mathcal{B}\mathcal{F}^2 + \mathcal{C}\mathcal{G}^2 - \mathcal{A}\mathcal{B}\mathcal{C} - 2\mathcal{E}\mathcal{F}\mathcal{G} = AE^2 + BF^2 + CG^2 - ABC - 2EFG \\ 4) & \mathcal{A}'\mathcal{H}^2 + \mathcal{B}'\mathcal{J}^2 + \mathcal{C}'\mathcal{K}^2 + 2\mathcal{E}'\mathcal{J}\mathcal{K} + 2\mathcal{F}'\mathcal{K}\mathcal{H} + 2\mathcal{G}'\mathcal{H}\mathcal{J} + (\mathcal{A}\mathcal{E}^2 + \mathcal{B}\mathcal{F}^2 + \mathcal{C}\mathcal{G}^2 - \mathcal{A}\mathcal{B}\mathcal{C} - 2\mathcal{E}\mathcal{F}\mathcal{G})\mathcal{L} = A'H'^2 + B'J'^2 + C'K'^2 + 2E'JK + 2F'KH + 2G'HJ + (AE^2 + BF^2 + CG^2 - ABC - 2EFG)L. \end{aligned}$$

Demonstr. I. Cum axes coordinatarum $x, y, z; x', y', z'$ inter se sint perpendiculares oportet esse (2) $(x-p)^2 + (y-q)^2 + (z-r)^2 = x'^2 + y'^2 + z'^2$. Unde aequationum (1) ope statim prodeunt sequentes

$$(3) \begin{cases} \xi^2 + \eta^2 + \zeta^2 = 1, \xi'^2 + \eta'^2 + \zeta'^2 = 1, \xi''^2 + \eta''^2 + \zeta''^2 = 1 \\ \xi\xi'' + \eta\eta'' + \zeta\zeta'' = 0, \xi'\xi'' + \eta'\eta'' + \zeta'\zeta'' = 0, \xi\xi' + \eta\eta' + \zeta\zeta' = 0 \end{cases}$$

Quibus rite cum aeqq. (1) combinatis nanciscemur

$$(4) \begin{cases} x' = \xi(x-p) + \eta(y-q) + \zeta(z-r) \\ y' = \xi'(x-p) + \eta'(y-q) + \zeta'(z-r) \\ z' = \xi''(x-p) + \eta''(y-q) + \zeta''(z-r) \end{cases}$$

et hisce valoribus in aequatione (2) substitutis habebimus etiam

$$(5) \begin{cases} \xi^2 + \xi'^2 + \xi''^2 = 1, \eta^2 + \eta'^2 + \eta''^2 = 1, \zeta^2 + \zeta'^2 + \zeta''^2 = 1 \\ \eta\zeta + \eta'\zeta' + \eta''\zeta'' = 0, \xi\zeta + \xi'\zeta' + \xi''\zeta'' = 0, \xi\eta + \xi'\eta' + \xi''\eta'' = 0 \end{cases}$$

Ex aeqq. (4), $\xi'\eta''\zeta'' - \zeta'\eta'' + \xi'(\eta''\zeta - \zeta''\eta) + \xi''(\eta\zeta' - \zeta\eta')$ per K denotando, derivantur:

$$K(x - p) = (\eta'\zeta'' - \zeta'\eta'')x' + (\eta''\zeta - \zeta''\eta)y' + (\eta\zeta' - \zeta\eta')z'$$

$$K(y - q) = (\zeta'\xi'' - \xi'\zeta'')x' + (\zeta''\xi - \xi''\zeta)y' + (\zeta\zeta' - \xi\xi')z'$$

$$K(z - r) = (\xi'\eta'' - \eta'\xi'')x' + (\xi''\eta - \eta''\xi)y' + (\xi\eta' - \eta\xi')z'$$

quae cum aeqq. (1) comparatae suppeditant:

$$(6) \begin{cases} \eta'\zeta' - \zeta'\eta'' = K\xi, & \eta''\zeta - \zeta''\eta = K\xi', & \eta\zeta' - \zeta\eta' = K\xi'', \\ \zeta'\xi'' - \xi'\zeta'' = K\eta, & \zeta''\xi - \xi''\zeta = K\eta', & \zeta\zeta' - \xi\xi' = K\eta'', \\ \xi'\eta'' - \eta'\xi'' = K\zeta, & \xi''\eta - \eta''\xi = K\zeta', & \xi\eta' - \eta\xi' = K\zeta'', \end{cases}$$

Unde sequitur $K^2 = (\eta'\zeta' - \zeta'\eta'')^2 + (\eta''\zeta - \zeta''\eta)^2 + (\eta\zeta' - \zeta\eta')^2 = (\eta^2 + \eta'^2 + \eta''^2)(\zeta^2 + \zeta'^2 + \zeta''^2) - (\eta\zeta + \eta'\zeta' + \eta''\zeta'')^2 = 1$ adeoque $K = \pm 1$. Ad decernendum, utrum signorum eligendum sit, ponamus $x - p = x'$, $y - q = y'$, $z - r = z'$, erit (1) $\xi = \eta' = \xi'' = 1$ et reliquae quantitates $\xi', \xi'', \eta, \eta'', \zeta, \zeta' = 0$. Quibus valoribus in $\eta'\zeta' - \zeta'\eta'' = K\xi$ substitutis erit $K = 1$.

II. Aequationem per substitutionem (1) ex (P) prodeuntem cum (Q) comparando statuendoque,

$$(7) \begin{cases} A\xi + G\eta + F\zeta = P, & A\xi' + G\eta' + F\zeta' = P', & A\xi'' + G\eta'' + F\zeta'' = P'', \\ B\eta + E\zeta + G\xi = Q, & B\eta' + E\zeta' + G\xi' = Q', & B\eta'' + E\zeta'' + G\xi'' = Q'', \\ C\zeta + F\xi + E\eta = R, & C\zeta' + F\xi' + E\eta' = R', & C\zeta'' + F\xi'' + E\eta'' = R'' \end{cases}$$

habebimus

$$(8) \begin{cases} \mathfrak{A} = P\xi + Q\eta + R\zeta, \\ \mathfrak{B} = P'\xi' + Q'\eta' + R'\zeta', & \mathfrak{C} = P''\xi'' + Q''\eta'' + R''\zeta'' \\ \mathfrak{E} = P'\xi'' + Q'\eta'' + R'\zeta'' = P''\xi' + Q''\eta' + R''\zeta' \\ \mathfrak{F} = P''\xi + Q''\eta + R\zeta'' = P\xi'' + Q\eta'' + R\zeta' \\ \mathfrak{G} = P\xi' + Q\eta' + R\zeta' = P'\xi + Q'\eta + R\zeta \\ \mathfrak{H} = Pp + Qq + Rr + H\xi + J\eta + K\zeta \\ \mathfrak{I} = P'p + Q'q + R'r + H\xi' + J\eta' + K\zeta' \\ \mathfrak{K} = P''p + Q''q + R''r + H\xi'' + J\eta'' + K\zeta'' \end{cases}$$

(10) $\xi = Ap^2 + Bq^2 + Cr^2 + 2Eqr + 2Frp + 2Gpq + 2Hp + 2Jq + 2Kr + L$.
Nec non ex aeqq. (7) deducuntur

$$(11) \begin{cases} A = P\xi + P'\xi' + P''\xi'', \\ B = Q\eta + Q'\eta' + Q''\eta'', & C = R\xi + R'\xi' + R''\xi'' \\ E = Q\xi + Q'\xi' + Q''\xi'' = R\eta + R'\eta' + R''\eta'' \\ F = R\xi + R'\xi' + R''\xi'' = P\xi + P'\xi' + P''\xi'' \\ G = P\eta + P'\eta' + P''\eta'' = Q\xi + Q'\xi' + Q''\xi'' \end{cases}$$

Ex primis tribus aequationum (8) (11) sua sponte sequitur prima aequatio conditionalis demonstranda $\mathcal{A} + \mathcal{B} + \mathcal{C} = A + B + C$.

III. Statuendo

$$(12) \begin{cases} Q'R'' - R'Q'' = \mathfrak{P}, & R'P'' - P'R'' = \mathfrak{Q}, & P'Q'' - Q'P'' = \mathfrak{R} \\ Q''R - R''Q = \mathfrak{P}', & R''P - P''R = \mathfrak{Q}', & P''Q - Q''P = \mathfrak{R}' \\ QR' - RQ' = \mathfrak{P}'', & RP' - PR' = \mathfrak{Q}'', & PQ' - QP' = \mathfrak{R}'' \end{cases}$$

adipiscemur, quantitatibus $\mathcal{A}'$, $\mathcal{B}'$, etc. per aeqq. (8) rite evolutis et aequationibus (6) adhibitis, $\mathcal{A}' = \mathfrak{P}\xi + \mathfrak{Q}\eta + \mathfrak{R}\xi$

$$(13) \begin{cases} \mathcal{B}' = \mathfrak{P}'\xi' + \mathfrak{Q}'\eta' + \mathfrak{R}'\xi', & \mathcal{C}' = \mathfrak{P}''\xi'' + \mathfrak{Q}''\eta'' + \mathfrak{R}''\xi'' \\ \mathcal{E}' = \mathfrak{P}'\xi'' + \mathfrak{Q}'\eta'' + \mathfrak{R}'\xi' = \mathfrak{P}''\xi + \mathfrak{Q}''\eta + \mathfrak{R}''\xi \\ \mathcal{F}' = \mathfrak{P}''\xi + \mathfrak{Q}''\eta + \mathfrak{R}''\xi = \mathfrak{P}'\xi' + \mathfrak{Q}'\eta' + \mathfrak{R}'\xi' \\ \mathcal{G}' = \mathfrak{P}\xi + \mathfrak{Q}\eta + \mathfrak{R}\xi = \mathfrak{P}'\xi + \mathfrak{Q}'\eta + \mathfrak{R}'\xi \end{cases}$$

nec non ex aequationibus (11) (6)

$$(14) \begin{cases} A' = \mathfrak{P}\xi + \mathfrak{P}'\xi' + \mathfrak{P}''\xi'' \\ B' = \mathfrak{Q}\eta + \mathfrak{Q}'\eta' + \mathfrak{Q}''\eta'', & C' = \mathfrak{R}\xi + \mathfrak{R}'\xi' + \mathfrak{R}''\xi'' \\ E' = \mathfrak{Q}\xi + \mathfrak{Q}'\xi' + \mathfrak{Q}''\xi'' = \mathfrak{R}\eta + \mathfrak{R}'\eta' + \mathfrak{R}''\eta'' \\ F' = \mathfrak{R}\xi + \mathfrak{R}'\xi' + \mathfrak{R}''\xi'' = \mathfrak{P}\xi + \mathfrak{P}'\xi' + \mathfrak{P}''\xi'' \\ G' = \mathfrak{P}\eta + \mathfrak{P}'\eta' + \mathfrak{P}''\eta'' = \mathfrak{Q}\xi + \mathfrak{Q}'\xi' + \mathfrak{Q}''\xi'' \end{cases}$$

Ex primis tribus harum aequationum (13) (14) statim evadit secunda aequatio conditionalis demonstranda $\mathcal{A}' + \mathcal{B}' + \mathcal{C}' = A' + B' + C'$.

Denotando $ABC + 2EFG - AE^2 - BF^2 - CG^2$ per $\mathfrak{D}$ et $\mathcal{ABC} + 2\mathcal{EFG} - \mathcal{A}\mathcal{E}^2 - \mathcal{B}\mathcal{F}^2 - \mathcal{C}\mathcal{G}^2$ per $\mathfrak{D}$, erit

$D = AA' + GG' + FF' = (11) (14) P\mathfrak{P} + Q\Omega + R\mathfrak{R}$
 $\mathfrak{D} = \mathfrak{A}\mathfrak{A}' + \mathfrak{G}\mathfrak{G}' + \mathfrak{F}\mathfrak{F}' = (8) (13) P\mathfrak{P} + P'\mathfrak{P}' + P''\mathfrak{P}''$
 Est autem, propter (12), $P\mathfrak{P} + Q\Omega + R\mathfrak{R} = P\mathfrak{P} + P'\mathfrak{P}' + P''\mathfrak{P}''$
 adeoque erit $D = \mathfrak{D}$, quae est secunda aequatio demonstranda.

IV. Quodsi jam statuimus

(15) $\mathfrak{H}\xi + \mathfrak{J}\xi' + \mathfrak{K}\xi'' = \mathfrak{H}'$, $\mathfrak{H}\eta + \mathfrak{J}\eta' + \mathfrak{K}\eta'' = \mathfrak{J}'$, $\mathfrak{H}\zeta + \mathfrak{J}\zeta' + \mathfrak{K}\zeta'' = \mathfrak{K}'$
 erit propter (9) (11) $\mathfrak{H}' = Ap + Gq + Fr + H$,
 $\mathfrak{J}' = Gp + Bq + Er + J$, $\mathfrak{K}' = Fp + Eq + Cr + K$ e quibus sequitur esse

$$(16) \begin{cases} Dp = A'\mathfrak{H}' + G'J' + F'\mathfrak{K}' - (A'H + G'J + F'K) \\ Dq = G'\mathfrak{H}' + B'J' + E'\mathfrak{K}' - (G'H + B'J + E'K) \\ Dr = F'\mathfrak{H}' + E'J' + C'\mathfrak{K}' - (F'H + E'J + C'K) \end{cases}$$

Est autem (10) $\mathfrak{L} - L = (Ap + Gq + Fr + 2H)p + (Gp + Bq + Er + 2J)q + (Fp + Eq + Cr + 2K)r = (\mathfrak{H}' + H)p + (\mathfrak{J}' + J)q + (\mathfrak{K}' + K)r$
 adeoque $D(\mathfrak{L} - L) = A'\mathfrak{H}'^2 + B'J'^2 + C'\mathfrak{K}'^2 + 2E'J'\mathfrak{K}' + 2F'\mathfrak{K}'\mathfrak{H}' + 2G'\mathfrak{H}'J'$
 $- (A'H^2 + B'J^2 + C'K^2 + 2E'JK + 2F'KH + 2G'HJ)$

Jam ex aeqq. (14) nullo negotio deducuntur

$$(17) \begin{cases} A'\xi + G'\eta + F'\zeta = \mathfrak{P}, A'\xi' + G'\eta' + F'\zeta' = \mathfrak{P}', A'\xi'' + G'\eta'' + F'\zeta'' = \mathfrak{P}'', \\ G'\xi + B'\eta + E'\zeta = \mathfrak{Q}, G'\xi' + B'\eta' + E'\zeta' = \mathfrak{Q}', G'\xi'' + B'\eta'' + E'\zeta'' = \mathfrak{Q}'', \\ F'\xi + E'\eta + C'\zeta = \mathfrak{R}, F'\xi' + E'\eta' + C'\zeta' = \mathfrak{R}', F'\xi'' + E'\eta'' + C'\zeta'' = \mathfrak{R}'', \end{cases}$$

unde ope aeqq. (13) derivantur

$$\begin{aligned} \mathfrak{A}' &= A'\xi^2 + B'\eta^2 + C'\zeta^2 + 2E'\eta\zeta + 2F'\xi\zeta + 2G'\xi\eta \\ \mathfrak{B}' &= A'\xi'^2 + B'\eta'^2 + C'\zeta'^2 + 2E'\eta'\zeta' + 2F'\xi'\zeta' + 2G'\xi'\eta' \\ \mathfrak{C}' &= A'\xi''^2 + B'\eta''^2 + C'\zeta''^2 + 2E'\eta''\zeta'' + 2F'\xi''\zeta'' + 2G'\xi''\eta'' \\ \mathfrak{D}' &= A'\xi'\xi'' + B'\eta'\eta'' + C'\zeta'\zeta'' + E'(\eta'\zeta' + \zeta''\eta'') + F'(\xi'\zeta' + \xi''\zeta'') + G'(\xi'\eta'' + \eta'\xi'') \\ \mathfrak{E}' &= A'\xi'\xi + B'\eta'\eta + C'\zeta'\zeta + E'(\eta'\zeta + \zeta''\eta) + F'(\xi'\zeta + \xi''\zeta) + G'(\xi'\eta + \eta'\xi) \\ \mathfrak{F}' &= A'\xi\xi' + B'\eta\eta' + C'\zeta\zeta' + E'(\eta\zeta + \zeta'\eta') + F'(\xi\zeta + \xi'\zeta') + G'(\xi\eta + \eta'\xi') \end{aligned}$$

Itaque valoribus ipsarum $\mathfrak{H}'$, $\mathfrak{J}'$, $\mathfrak{K}'$ (15) substitutis, habebitur

$$D(L-1) = \mathcal{A}h^2 + \mathcal{B}j^2 + \mathcal{C}k^2 + 2\mathcal{C}j\mathcal{K} + 2\mathcal{B}k\mathcal{H} + 2\mathcal{G}h\mathcal{J} \\ = A'H^2 + B'J^2 + C'K^2 + 2E'JK + 2F'KH + 2G'HJ$$

quae manifesto est quarta aequatio conditionalis demonstranda.

§. 6.

Problema. Aequationem generalem (P) $Ax^2 + By^2 + Cz^2 + 2Eyz + 2Fzx + 2Gxy + 2Hx + 2Jy + 2Kz + L = 0$ in simpliciorum (R) $\mathcal{A}x'^2 + \mathcal{B}y'^2 + \mathcal{C}z'^2 + 2\mathcal{H}x = 0$ transformare.

Solutio. Propter aequationes conditionales, quae inter coefficients aequationum (P) (R) locum habere debent, oportet ut sit: $\mathcal{A} + \mathcal{B} + \mathcal{C} = A + B + C$, $\mathcal{B}\mathcal{C} + \mathcal{C}\mathcal{A} + \mathcal{A}\mathcal{B} = A' + B' + C'$, $\mathcal{A}\mathcal{B}\mathcal{C} = ABC + 2EFG - AE^2 - BF^2 - CG^2$, et $\mathcal{B}\mathcal{C}h^2 = A'H^2 + B'J^2 + C'K^2 + 2E'JK + 2F'KH + 2G'HJ + L$ Ex primis tribus aequationibus colligimus fore $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ tres radices aequationis cubicae

(a) $t^3 - (A+B+C)t^2 + (A'+B'+C')t + AE^2 + BF^2 + CG^2 - ABC - 2EFG = 0$, quarum certe unam realem esse constat. Ponamus hanc radicem esse $\mathcal{A}$, habebitur, dividendo aequationem (a) per $t - \mathcal{A}$, aequatio $t^2 - 2mt + n = 0$, cuius coefficients $2m = \mathcal{B} + \mathcal{C}$, $n = \mathcal{B}\mathcal{C}$ quantitates reales esse debent. Aequatio (a) etiam hoc modo exhiberi potest

$$(t - A)(t - B)(t - C) - E^2(t - A) - F^2(t - B) - G^2(t - C) - 2EFG = 0, \\ \text{unde } t - A = \frac{F^2(t - B) + G^2(t - C) + 2EFG}{(t - B)(t - C) - E^2} \text{ adeoque}$$

$$\mathcal{B} - A = \frac{F^2(\mathcal{B} - B) + G^2(\mathcal{B} - C) + 2EFG}{(\mathcal{B} - B)(\mathcal{B} - C) - E^2}, \mathcal{C} - A = \frac{F^2(\mathcal{C} - B) + G^2(\mathcal{C} - C) + 2EFG}{(\mathcal{C} - B)(\mathcal{C} - C) - E^2},$$

quarum secunda a prima subtracta et residuo per $\mathcal{C} - \mathcal{B}$ diviso nanciscemur

$$-1 = \frac{F^2(B-B)(C-B) + G^2(B-C)(C-C) + 2EFG(B+C-B-C) + E^2(F+G)}{[(B-B)(B-C) - E][(C-B)(C-C) - E]}$$

$$= \frac{(F+B)(F+C) + (G+B)(G+C) - (F+mF)^2 + (G+mG)^2 + (n-m^2)(E^2+F^2)}{[B^2 - (B+C)B + A][C^2 - (B+C)C + A] - [n-m(B+C) + A]^2 + [(B-C)^2 + 4E^2](n-m^2)}$$

cujus fractionis valor, si $n - m^2$ est positiva, etiam positivum erit adeoque $= -1$ esse nequit. Itaque oportet ut sit $m^2 - n$ positiva, unde concludimus, radices duas aequationis $t^2 - 2mt + n = 0$, adeoque omnes tres radices aequationis cubicae (a) fore reales.

II. Quodsi hae tres radices non omnes sint positivae, $\mathcal{A}$ semper ita assumi licet, ut $\mathcal{B}\mathcal{C}$ adeoque $\mathcal{B}\mathcal{C}\mathcal{H}^2$ ad lubitum sit quantitas positiva seu negativa; sin autem omnes tres radices positivae sunt $\mathcal{B}\mathcal{C}\mathcal{H}^2$ erit semper positiva. Itaque aequatio (P) semper in simpliciores (R) transmutare licet, nisi simul $A + B + C$, $A' + B' + C'$, $ABC + 2EFH - (AE^2 + BF^2 + CG^2)$, sint quantitates positivae et $A'H^2 + B'J^2 + C'K^2 + 2E'JK + 2F'KH + 2G'HJ + L$ quantitas negativa.

III. Ex aeqq. (8) (§. 5) nullo negotio deducuntur

$$(b) \begin{cases} P = \mathcal{A}\xi + \mathcal{B}\xi' + \mathcal{C}\xi'', P' = \mathcal{B}\xi + \mathcal{C}\xi' + \mathcal{A}\xi, P'' = \mathcal{C}\xi + \mathcal{A}\xi' + \mathcal{B}\xi'' \\ Q = \mathcal{A}\eta + \mathcal{B}\eta' + \mathcal{C}\eta'', Q' = \mathcal{B}\eta + \mathcal{C}\eta' + \mathcal{A}\eta, Q'' = \mathcal{C}\eta + \mathcal{A}\eta' + \mathcal{B}\eta'' \\ R = \mathcal{A}\zeta + \mathcal{B}\zeta' + \mathcal{C}\zeta'', R' = \mathcal{B}\zeta + \mathcal{C}\zeta' + \mathcal{A}\zeta, R'' = \mathcal{C}\zeta + \mathcal{A}\zeta' + \mathcal{B}\zeta'' \end{cases}$$

unde adjumento (7) (§. 5), statuendo $\mathcal{E}, \mathcal{F}, \mathcal{G} = 0$ nanciscemur

$$(A-\mathcal{A})\xi + G\eta + F\zeta = 0, (A-\mathcal{B})\xi' + G\eta' + F\zeta' = 0, (A-\mathcal{C})\xi'' + G\eta'' + F\zeta'' = 0 \\ G\xi + (B-\mathcal{A})\eta + E\zeta = 0, G\xi' + (B-\mathcal{B})\eta' + E\zeta' = 0, G\xi'' + (B-\mathcal{C})\eta'' + E\zeta'' = 0 \\ F\xi + E\eta + (C-\mathcal{A})\zeta = 0, F\xi' + E\eta' + (C-\mathcal{B})\zeta' = 0, F\xi'' + E\eta'' + (C-\mathcal{C})\zeta'' = 0$$

e quibus aequationibus colligitur esse

$$\xi : \eta : \zeta = [(B-\mathcal{A})(C-\mathcal{A}) - E] : [EF - G(C-\mathcal{A})] : [GE - F(B-\mathcal{A})] \\ = [EF - G(C-\mathcal{A})] : [(C-\mathcal{A})(A-\mathcal{A}) - F] : [FG - E(A-\mathcal{A})] \\ = [GE - F(B-\mathcal{A})] : [FG - E(A-\mathcal{A})] : [(A-\mathcal{A})(B-\mathcal{A}) - G]$$

$$\begin{aligned}
\xi:\eta:\zeta &= [(B-\mathfrak{B})(C-\mathfrak{B})-E^2]: [EF-G(C-\mathfrak{B})]: [GE-F(B-\mathfrak{B})] \\
&= [EF-G(C-\mathfrak{B})]: [(C-\mathfrak{B})(A-\mathfrak{B})-F^2]: [FG-E(A-\mathfrak{B})] \\
&= [GE-F(B-\mathfrak{B})]: [FG-E(A-\mathfrak{B})]: [(A-\mathfrak{B})(B-\mathfrak{B})-G^2] \\
\xi'':\eta'':\zeta'' &= [(B-\mathfrak{C})(C-\mathfrak{C})-E^2]: [EF-G(C-\mathfrak{C})]: [GE-F(B-\mathfrak{C})] \\
&= [EF-G(C-\mathfrak{C})]: [(C-\mathfrak{C})(A-\mathfrak{C})-F^2]: [FG-E(A-\mathfrak{C})] \\
&= [GE-F(B-\mathfrak{C})]: [FG-E(A-\mathfrak{C})]: [(A-\mathfrak{C})(B-\mathfrak{C})-G^2].
\end{aligned}$$

Atque ex hisce sequitur esse

$$\begin{aligned}
\xi^2:\eta^2:\zeta^2 &= [(B-\mathfrak{A})(C-\mathfrak{A})-E^2]: [(C-\mathfrak{A})(A-\mathfrak{A})-F^2]: [(A-\mathfrak{A})(B-\mathfrak{A})-G^2] \\
\xi'^2:\eta'^2:\zeta'^2 &= [(B-\mathfrak{B})(C-\mathfrak{B})-E^2]: [(C-\mathfrak{B})(A-\mathfrak{B})-F^2]: [(A-\mathfrak{B})(B-\mathfrak{B})-G^2] \\
\xi''^2:\eta''^2:\zeta''^2 &= [(B-\mathfrak{C})(C-\mathfrak{C})-E^2]: [(C-\mathfrak{C})(A-\mathfrak{C})-F^2]: [(A-\mathfrak{C})(B-\mathfrak{C})-G^2] \\
\text{Est autem } (B-\mathfrak{A})(C-\mathfrak{A})-E^2 + (C-\mathfrak{A})(A-\mathfrak{A})-F^2 + (A-\mathfrak{A})(B-\mathfrak{A})-G^2 \\
&= A'+B'+C'-2\mathfrak{A}(A+B+C)+3\mathfrak{A}^2 = \mathfrak{B}\mathfrak{C}+\mathfrak{C}\mathfrak{A}+\mathfrak{A}\mathfrak{B}-2\mathfrak{A}(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}) \\
&+3\mathfrak{A}^2 = \mathfrak{B}\mathfrak{C}-\mathfrak{A}\mathfrak{C}-\mathfrak{A}\mathfrak{B}+\mathfrak{A}^2 = (\mathfrak{B}-\mathfrak{A})(\mathfrak{C}-\mathfrak{A})
\end{aligned}$$

Simili modo invenietur

$$\begin{aligned}
(B-\mathfrak{B})(C-\mathfrak{B})-E^2 + (C-\mathfrak{B})(A-\mathfrak{B})-F^2 + (A-\mathfrak{B})(B-\mathfrak{B})-G^2 &= (\mathfrak{C}-\mathfrak{B})(\mathfrak{A}-\mathfrak{B}) \\
(B-\mathfrak{C})(C-\mathfrak{C})-E^2 + (C-\mathfrak{C})(A-\mathfrak{C})-F^2 + (A-\mathfrak{C})(B-\mathfrak{C})-G^2 &= (\mathfrak{A}-\mathfrak{C})(\mathfrak{B}-\mathfrak{C})
\end{aligned}$$

Adeoque

$$\begin{aligned}
\xi &= \sqrt{\frac{(B-\mathfrak{A})(C-\mathfrak{A})-E^2}{(B-\mathfrak{A})(\mathfrak{C}-\mathfrak{A})}}, \eta = \sqrt{\frac{(C-\mathfrak{A})(A-\mathfrak{A})-F^2}{(B-\mathfrak{A})(\mathfrak{C}-\mathfrak{A})}}, \zeta = \sqrt{\frac{(A-\mathfrak{A})(B-\mathfrak{A})-G^2}{(B-\mathfrak{A})(\mathfrak{C}-\mathfrak{A})}} \\
\xi' &= \sqrt{\frac{(B-\mathfrak{B})(C-\mathfrak{B})-E^2}{(\mathfrak{C}-\mathfrak{B})(\mathfrak{A}-\mathfrak{B})}}, \eta' = \sqrt{\frac{(C-\mathfrak{B})(A-\mathfrak{B})-F^2}{(\mathfrak{C}-\mathfrak{B})(\mathfrak{A}-\mathfrak{B})}}, \zeta' = \sqrt{\frac{(A-\mathfrak{B})(B-\mathfrak{B})-G^2}{(\mathfrak{C}-\mathfrak{B})(\mathfrak{A}-\mathfrak{B})}} \\
\xi'' &= \sqrt{\frac{(B-\mathfrak{C})(C-\mathfrak{C})-E^2}{(\mathfrak{A}-\mathfrak{C})(\mathfrak{B}-\mathfrak{C})}}, \eta'' = \sqrt{\frac{(C-\mathfrak{C})(A-\mathfrak{C})-F^2}{(\mathfrak{A}-\mathfrak{C})(\mathfrak{B}-\mathfrak{C})}}, \zeta'' = \sqrt{\frac{(A-\mathfrak{C})(B-\mathfrak{C})-G^2}{(\mathfrak{A}-\mathfrak{C})(\mathfrak{B}-\mathfrak{C})}}
\end{aligned}$$

Quia rationes $\xi:\eta:\zeta$; $\xi':\eta':\zeta'$; $\xi'':\eta'':\zeta''$ sunt reales, necesse est propter aeqq. (2) (§. 5), ut quantitates ξ , η , ζ , etc. ipsae reales sint.

IV. Aequationes (15) (§. 5) statuendo $\mathfrak{J}$, $\mathfrak{K} = 0$ fiunt

$$\begin{aligned}
Dp &= (A'\xi + G'\eta + F'\zeta)\mathfrak{H} - (A'H + G'J + F'K) \\
Dq &= (B'\eta + E'\zeta + G'\xi)\mathfrak{H} - (B'J + E'K + H'G) \\
Dr &= (C'\zeta + F'\xi + E'\eta)\mathfrak{H} - (C'K + F'H + E'J)
\end{aligned}$$

Ex aeqq. (17) (15) (§. 5) deducitur esse

$$A'\xi + G'\eta + F'\zeta = \mathfrak{P} = \mathfrak{A}'\xi + \mathfrak{G}'\xi' + \mathfrak{F}'\xi''$$

$$B'\eta + E'\zeta + G'\zeta = \mathfrak{Q} = \mathfrak{A}'\eta + \mathfrak{G}'\eta' + \mathfrak{F}'\eta''$$

$$C'\zeta + F'\xi + E'\eta = \mathfrak{R} = \mathfrak{A}'\zeta + \mathfrak{G}'\zeta' + \mathfrak{F}'\zeta''$$

et quia, existente $\mathfrak{E}, \mathfrak{F}, \mathfrak{G} = 0$, $\mathfrak{G}' = 0$, $\mathfrak{F}' = 0$, $\mathfrak{A}' = \mathfrak{B}\mathfrak{C}$ et $D = \mathfrak{A}\mathfrak{B}\mathfrak{C}$

habebimus $\mathfrak{A}p = \mathfrak{H}\xi - \frac{1}{2\mathfrak{B}\mathfrak{C}} (A'H + G'J + F'K)$

$$\mathfrak{A}q = \mathfrak{H}\eta - \frac{1}{2\mathfrak{B}\mathfrak{C}} (B'J + E'K + G'H)$$

$$\mathfrak{A}r = \mathfrak{H}\zeta - \frac{1}{2\mathfrak{B}\mathfrak{C}} (C'K + F'H + E'J)$$

V. Ceterum nullo negotio significatio geometrica quantitatum ξ, η, ζ etc. p, q, r assignari licet. Statuamus enim XO, YO, ZO esse tres axes coordinatarum x, y, z et $X'O, Y'O, Z'O$ lineas axibus x', y', z' parallelas per punctum intersectionis O axium x, y, z positas, erunt $\xi, \eta, \zeta; \xi', \eta', \zeta'; \xi'', \eta'', \zeta''$ respective cosinus angulorum $X'OX, X'OY, X'OZ; Y'OX, Y'OY, Y'OZ; Z'OX, Z'OY, Z'OZ$ et p, q, r coordinatae puncti intersectionis axium x', y', z' respectu axium x, y, z .

§. 7.

Coroll. Si D non est $= 0$, aequatio generalis (P) etiam in formam $\mathfrak{A}x^2 + \mathfrak{B}y^2 + \mathfrak{C}z^2 + \mathfrak{L} = 0$ redigi potest. Quo in casu pro quantitibus $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{L}, \xi, \eta, \zeta$ etc. iidem valores ut antea invenientur. Reliquae autem quantitates sequenti modo determinantur.

$$-Dp = A'H + G'J + F'K; -Dq = B'J + E'K + G'H; -Dr = C'K + F'G + E'J$$

$$\text{et } D\mathfrak{L} = A'H^2 + B'J^2 + C'K^2 + 2E'JK + 2F'KH + 2G'HJ - LD.$$

§. 8.

Coroll. Si aequatio $Ax^2 + By^2 + Cz^2 + 2Eyz + 2Fzx + 2Gxy + 2Hx + 2Jy + 2Kz$ per substitutionem $x = \xi x' + \xi' y' + \xi'' z'$;

$y = \eta x' + \eta' y' + \eta'' z'$; $z = \zeta x' + \zeta' y' + \zeta'' z'$ in similem
 $\mathcal{A}x'^2 + \mathcal{B}y'^2 + \mathcal{C}z'^2 + 2\mathcal{E}y'z' + 2\mathcal{F}z'x' + 2\mathcal{G}x'y' + 2\mathfrak{H}x' + 2\mathfrak{J}y' + 2\mathfrak{K}z' = 0$
 transmutatur, ad quatuor aequationes conditionales (§. 5) jam in-
 ventos adhuc duae adjiciendae sunt. Ex aeqq. (9) (§. 5) enim colligitur
 esse $\mathfrak{H} = H\zeta + \mathfrak{J}\eta + K\zeta$; $\mathfrak{J} = H\zeta' + \mathfrak{J}\eta' + K\zeta'$; $\mathfrak{K} = H\zeta'' + \mathfrak{J}\zeta''$
 $+ \mathfrak{K}\zeta''$, unde sua sponte demanat 5) $\mathfrak{H}^2 + \mathfrak{J}^2 + \mathfrak{K}^2 = H^2 + J^2 + K^2$
 nec non $H = \zeta\mathfrak{H} + \zeta'\mathfrak{J} + \zeta''\mathfrak{K}$; $J = \eta\mathfrak{H} + \eta'\mathfrak{J} + \eta''\mathfrak{K}$; $K = \zeta\mathfrak{H} + \zeta'\mathfrak{J} + \zeta''\mathfrak{K}$;
 quibus valoribus substitutis, fit 6) $AH^2 + BJ^2 + CK^2 + 2EJK$
 $+ 2FKH + 2GHJ = \mathcal{A}\mathfrak{H}^2 + \mathcal{B}\mathfrak{J}^2 + \mathcal{C}\mathfrak{K}^2 + 2\mathcal{E}\mathfrak{J}\mathfrak{K} + 2\mathcal{F}\mathfrak{K}\mathfrak{H} + 2\mathcal{G}\mathfrak{H}\mathfrak{J}$.
 Unde patet, novem quantitatum $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ etc. tantummodo tres
 ad libitum determinari posse. Ponamus e. g. $\mathcal{C} = 0$, $\mathfrak{J} = 0$, $\mathfrak{K} = 0$,
 habebimus aequationes

- 1) $\mathcal{A} + \mathcal{B} + \mathcal{C} = A + B + C$; 2) $\mathcal{B}\mathcal{C} + \mathcal{C}\mathcal{A} + \mathcal{A}\mathcal{B} - \mathfrak{F}^2 - \mathfrak{G}^2 = A' + B' + C'$
 - 3) $\mathcal{B}\mathfrak{J}^2 + \mathcal{C}\mathfrak{G}^2 - \mathcal{A}\mathcal{B}\mathcal{C} = AE^2 + BF^2 + CG^2 - ABC - 2EFG$
 - 4) $\mathcal{B}\mathcal{C}\mathfrak{H}^2 = A'H^2 + B'J^2 + C'K^2 + 2E'JK + 2F'KH + 2G'HJ$
 - 5) $\mathfrak{H}^2 = H^2 + J^2 + K^2$; 6) $\mathcal{A}\mathfrak{H}^2 = AH^2 + BJ^2 + CK^2 + 2EJK + 2FKH + 2GHJ$
- e quibus, quantitibus A , B , C , etc. datis, nullo negotio $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$,
 $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ determinari possunt.

§. 9.

Coronidis loco huic disquisitioni quaedam de curvarum et su-
 perficierum tangentibus et circulis curvaturae subjungemus, suppo-
 nendo tamquam demonstrata sequentia theoremata.

I. Tangens curvae planae, cujus aequatio est $2px = ay^2$
 $+ a'y^3 + a''y^4 + \text{etc.}$, pro puncto, cujus $y = 0$, est axis coordi-
 natarum y et radius osculi pro hoc puncto $= \frac{p}{a}$

II. Superficie, cujus aequatio est $2px = \mathcal{B}y^2 + \mathcal{C}z^2 + \Omega$,
 Ω functionem ipsarum y , z altioris quam secundae dimensionis

denotante, planum tangens pro puncto, cujus y et $z = 0$, est planum axium y, z .

III. Sint $2px = ay^2 + a'y^3 + a''y^4 + \text{etc.}$ et $z = b'y^3 + b''y^4 + \text{etc.}$ aequationes cujuslibet curvae, tangens ejus pro puncto, cujus $y = 0$, erit axis coordinatarum y et planum circuli curvaturae ejus, cujus radius est $= \frac{P}{a}$, cum plano axium xy congruet.

§. 10.

Problema. Aequatione curvae planae $\phi(u, v) = 0$ data, pro quocunque ejus puncto tangentem et radium osculi determinare.

Solutio. Ponamus aequationem $\phi(u, v) = 0$ substituendo $u + x, v + y$ loco ipsarum u, v transire in (a) $2Ex + 2Fy + Ax^2 + By^2 + 2Cxy + \Omega = 0$, Ω functionem ipsarum x, y superioris quam secundae dimensionis denotante, erit

1) $Ex + Fy = 0$ aequatio tangentis,

2) $\frac{(E^2 + F^2)^{\frac{3}{2}}}{AF^2 + BE^2 - 2CEF}$ radius circuli curvaturae et $\frac{(E^2 + F^2)E}{AF^2 + BE^2 - 2CEF}$,
 $\frac{(E^2 + F^2)F}{AF^2 + BE^2 - 2CEF}$ coordinatae centri hujus circuli respectu axium x, y .

Demonstr. Aequationem (a) per substitutionem

$$x = \frac{E}{\sqrt{E^2 + F^2}}x' + \frac{F}{\sqrt{E^2 + F^2}}y', \quad y = \frac{F}{\sqrt{E^2 + F^2}}x' - \frac{E}{\sqrt{E^2 + F^2}}y'$$

in $2Ex' + Ax'^2 + By'^2 + 2Cx'y' + \Omega = 0$ transmutari ex (§. 4) apparet, quam per $-2Ex = By'^2 + b'y'^3 + b''y'^4 + \text{etc.}$ exhiberi posse nullo negotio perspicitur. Erit ergo (§. 1) axis coordinatarum y' , cujus aequatio $x' = 0$ aut $Ex + Fy = 0$, tangens

et $-\frac{C}{B} = (\S. 4) \frac{(E^2 + F^2)^{\frac{3}{2}}}{AF^2 + BE^2 - 2CEF}$ radius osculi. Quodsi jam statuimus, a, b esse coordinatas centri circuli osculi, erit, quia $Fx - Ey = 0$ est aequatio normalis, $Fa - Eb = 0$, quae aequatio cum $a^2 + b^2 = \frac{(E^2 + F^2)^3}{AF^2 + BE^2 - 2CEF}$ combinata suppeditat $a = \frac{(E^2 + F^2)E}{AF^2 + BE^2 - 2CEF}$,
 $b = \frac{(E^2 + F^2)F}{AF^2 + BE^2 - 2CEF}$.

Problema. Aequatione superficiei $\phi(u, v, w) = 0$ data, pro quocunque ejus puncto planum tangens et radium circuli curvaturae cujuslibet curvae ex intersectione superficiei cum plano normali ortae determinare.

Solutio. Ponamus aequationem $\phi(u, v, w) = 0$, substituendo $u + x, v + y, w + z$ respective loco ipsarum u, v, w , transire, in (b) $2Hx + 2Jy + 2Kz + Ax^2 + By^2 + Cz^2 + 2Eyz + 2Fzx + 2Gxy + \Omega = 0$, Ω functionem ipsarum x, y, z altioris quam secundae dimensionis denotante, erit aequatio plani tangentis superficiei pro puncto, cujus coordinatae u, v, w , $Hx + Jy + Kz = 0$; et statuendo brevitatis gratia $H^2 + J^2 + K^2 = T^2$ $(B+C)H^2 + (C+A)J^2 + (A+B)K^2 - 2EJK - 2FKH - 2GHJ = U$ nec non, denotationibus ipsarum A', B', C', E', F', G' (§. 5) salvis, $A'H^2 + B'J^2 + C'K^2 + 2E'JK + 2F'KH + 2G'HJ = V$, habebitur radius osculi cujusunque curvae normalis per punctum datum transeuntis $= \frac{-2T^3}{U + \cos 2\omega \sqrt{U^2 - 4VT^2}}$, ubi denotat ω angulum inclinationis ejus plani ad planum circuli maximi.

Demonstr. Aequationem (b) per substitutionem (§. 8) in $2\mathfrak{H}x' + \mathfrak{A}x'^2 + 2\mathfrak{B}y' + \mathfrak{C}z'^2 + 2\mathfrak{F}x'z' + 2\mathfrak{G}y'z' + \Omega = 0$, Ω denotan-

te functionem x', y', z majoris quam secundae dimensionis, transmutari posse, nullo negotio apparet, nec non perspicitur, hanc aequationem etiam ita per $-2\mathfrak{H}x' = \mathfrak{B}y'^2 + \mathfrak{C}z'^2 + \Omega'$, exhiberi posse, ut sit Ω' functio ipsarum y', z' altioris quam secundae dimensionis. Cum ergo (II §. 9) planum axium y', z' cujus aequatio $x' = 0$ aut respectu axium x, y, z , est $\xi x + \eta y + \zeta z = 0$, sit planum tangens quaesitum et propter aequationes $\xi'H + \eta'J + \zeta'K = 0$, $\xi''H + \eta''J + \zeta''K = 0$, habeatur $H:J:K = (\eta'\zeta'' - \zeta'\eta''):(\zeta'\xi'' - \xi'\zeta''):(\xi'\eta'' - \eta'\xi'') = \xi:\eta:\zeta$, erit $Hx + Jy + Kz = 0$ aequatio plani tangentis. Quia $\cos \omega y' + \sin \omega z' = 0$ est aequatio plani normalis sub angulo ω ad planum $x'z'$ inclinati, duae aequationes $-2\mathfrak{H}x' = \mathfrak{B}y'^2 + \mathfrak{C}z'^2 + \Omega'$, $\cos \omega y' + \sin \omega z' = 0$ exhibebunt indolem sectionis communis hujus plani et superficiei datae, quas per substitutionem $x' = x''$, $y' = y'' \cos \omega + z'' \sin \omega$, $z' = y'' \sin \omega - z'' \cos \omega$ in hasce $-2\mathfrak{H}x'' = \frac{1}{2}(\mathfrak{B} + \mathfrak{C} + (\mathfrak{B} - \mathfrak{C}) \cos 2\omega) y''^2 + \Omega$ et $z'' = 0$ transeunt. Erit ergo (§. I) radius osculi hujus curvae

$$= -\frac{2\mathfrak{H}^3}{\mathfrak{B} + \mathfrak{C} + (\mathfrak{B} - \mathfrak{C}) \cos 2\omega}, \text{ cujus maximum, existente } \mathfrak{B} < \mathfrak{C},$$

$$= -\frac{\mathfrak{H}^3}{\mathfrak{B}} \text{ et minimum } = -\frac{\mathfrak{H}^3}{\mathfrak{C}} \text{ esse, manifestum est. Est autem (§. 8)}$$

$$\mathfrak{H} = T = \sqrt{H^2 + J^2 + K^2}, \quad \mathfrak{B} + \mathfrak{C} = A + B + C - \mathfrak{A} = \frac{U}{T^2}$$

$$\mathfrak{B}\mathfrak{C} = \frac{V}{T^2} \text{ adeoque } \mathfrak{B} - \mathfrak{C} = \sqrt{(\mathfrak{B} + \mathfrak{C})^2 - 4\mathfrak{B}\mathfrak{C}} = \frac{\sqrt{U^2 - 4VT^2}}{T^2}$$

$$\text{et radius osculi} = \frac{2 - T^3}{U + \cos 2\omega \sqrt{U^2 - VT^2}} \text{ cujus maximum}$$

$$= \frac{-2T^3}{U + \sqrt{U^2 - VT^2}} \text{ et minimum} = \frac{-2T^3}{U - \sqrt{U^2 - VT^2}}$$

Problema. Lineae dupliciter curvatae tangentes et circulos osculi determinare.

Solutio. Denotando per u, v, w coordinatas cujus libet puncti curvae, quae tanquam functiones quantitatis mutabilis communis t considerentur, et per x, y, z harum coordinatarum differentias differentiae θ ipsius t respondentes, sit $x = \dot{u}\theta + \ddot{u}\theta^2 + \text{etc.}$, $y = \dot{v}\theta + \ddot{v}\theta^2 + \text{etc.}$, $z = \dot{w}\theta + \ddot{w}\theta^2 + \text{etc.}$ Hae tres aequationes per substitutionem $x = \xi x' + \eta y' + \zeta z'$, $y = \eta x' + \eta' y' + \eta'' z'$, $z = \zeta x' + \zeta' y' + \zeta'' z'$, statuendo (1) $\dot{u}\xi + \dot{v}\eta + \dot{w}\zeta = 0$, (2) $\dot{u}\xi'' + \dot{v}\eta'' + \dot{w}\zeta'' = 0$, (3) $\ddot{u}\xi'' + \ddot{v}\eta'' + \ddot{w}\zeta'' = 0$ permutantur in $x' = (\ddot{u}\xi + \ddot{v}\eta + \ddot{w}\zeta)\theta^2 + a''\theta^3 + \text{etc.}$, $y' = (\dot{u}\xi' + \dot{v}\eta' + \dot{w}\zeta')\theta + b'\theta^2 + \text{etc.}$, $z' = c''\theta^3 + c'''\theta^4 + \text{etc.}$, quae, eliminando θ , in has duas (4) $2(\dot{u}\xi' + \dot{v}\eta' + \dot{w}\zeta')^2 x' = (\ddot{u}\xi + \ddot{v}\eta + \ddot{w}\zeta)y'^2 + p y'^3 + \text{etc.}$ et $z' = q'' y'^3 + q''' y'^4 + \text{etc.}$ transeunt. Jam ex aeqq. (2) (3) designando $\dot{v}w - \dot{w}v$, $\dot{w}u - \dot{u}w$, $\dot{u}v - \dot{v}u$ respective per U, V, W et $\sqrt{U^2 + V^2 + W^2}$ per S , sequitur $\xi'' = \frac{U}{S}$, $\eta'' = \frac{V}{S}$, $\zeta'' = \frac{W}{S}$ et ex aeqq. (1), (2) $\dot{u} : \dot{v} : \dot{w} = \eta''\zeta' - \zeta''\eta : \zeta'\xi - \xi''\zeta : \xi''\eta - \eta''\xi = (\S. 5) (6) \xi' : \eta' : \zeta'$ adeoque denotando $\sqrt{\dot{u}^2 + \dot{v}^2 + \dot{w}^2}$ per s erit $\xi = \frac{\dot{u}}{s}$, $\eta = \frac{\dot{v}}{s}$, $\zeta = \frac{\dot{w}}{s}$. Nec non invenitur $\xi = \eta'\zeta'' - \zeta'\eta'' = \frac{\dot{v}W - \dot{w}V}{sS}$, $\eta = \zeta'\xi'' - \xi'\zeta'' = \frac{\dot{w}U - \dot{u}W}{sS}$, $\zeta = \xi'\eta'' - \eta'\xi'' = \frac{\dot{u}V - \dot{v}U}{sS}$ et $\ddot{u}\xi + \ddot{v}\eta + \ddot{w}\zeta = -\frac{S}{s}$. Aequationes (4) ergo fiunt $2s^2 x' = -\frac{S}{s} y'^2 + p y'^3$; $z' = q' y'^3 + q'' y'^4 + \text{etc.}$

Itaque tangens curvae, quae (III §. 9) est axis coordinatarum y' , ita determinatur, ut cosinus angulorum, quos cum axibus x, y, z format aut ξ, η, ζ sint in ratione $\dot{u} : \dot{v} : \dot{w}$; aequatio plani circuli osculi est $\xi x + \eta y + \zeta z = 0$ et denique radius osculi $= \frac{s^3}{S}$, unde simul ordinatae ejus centri respectu axium x, y, z , $\frac{s^2(\dot{v}W - \dot{w}V)}{S^2}$, $\frac{s^2(\dot{w}U - \dot{u}W)}{S^2}$, $\frac{s^2(\dot{u}V - \dot{v}U)}{S^2}$ evadunt.

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Disquisitiones quatuor ad theoriā functionum analyticarum pertinentes

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